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OSCILLATION OF SOLUTIONS OF A GENERALIZED LIÉNARD EQUATION

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ABSTRACT. The generalized Liénard equation, $\ddot{x}+f(x,\dot{x})+h(x)=0$, with $xh(x)>0$ and $yf(x,y)>0$ for nonzero x and y, is considered here, subject to the additional condition that $|f(x,y)|$ is not greater than $k(x)\,|y|^\alpha$ where α is a positive number and $k(x)$ is a continuous function which is positive for nonzero x. In case $\alpha\geqq 2$, all solutions of this Liénard equation are oscillatory. In case $0<\alpha<2$, sufficient conditions are given which insure that all solutions are oscillatory.

1. **Introduction.** We wish to consider the ordinary differential equation,

$$\ddot{x}+f(x,\dot{x})+h(x)=0 \tag{1.1}$$

a generalized Liénard equation, subject to the conditions:

(a) f and h are continuous in E_2 and E_1 respectively.

(b) $xh(x)>0$ for nonzero x.

(c) $yf(x,y)>0$ unless $x=0$ or $y=0$.

(d) There exists $\alpha>0$, $\varepsilon>0$ and a continuous function $k(x)$ defined on E_1 satisfying $k(x)>0$ for nonzero x such that $|f(x,y)|\leqq k(x)\,|y|^\alpha$ for all x, y satisfying $|x|, |y|<\varepsilon$.

Equation (1.1) governs nonlinear vibrations with nonlinear damping. This type of motion is particularly well known in case $f(x,y)=C\,|y|^\alpha \operatorname{sgn} y$, with $\alpha=0$, 1 or 2. (See [5, pp. 67–92].)

A solution $x(t)$ of (1.1) on an interval $[t_0,\infty)$ is said to oscillate at $+\infty$ if for any $t_1\geqq t_0$, $x(t)$ changes on sign on $[t_1,\infty)$. If every nontrivial solution oscillates, then the equation (1.1) is said to be oscillatory. In this paper sufficient conditions are given for (1.1) to be oscillatory.

Here it is found that if (1.1) satisfies (a)–(d) with $\alpha\geqq 2$, and if the zero solution of (1.1) is globally asymptotically stable, then (1.1) is oscillatory. In case $0<\alpha<2$, an integral inequality involving $h(x)$, $k(x)$ and α is a sufficient condition for oscillation. In §4, examples are given which show

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that this integral condition is sharp. Nonoscillatory equations are given for each α, $0<\alpha<2$, which are just on the boundary between oscillation and nonoscillation. These examples generalize the phenomenon of *critical damping*, well known for linear equations with constant coefficients, to certain nonlinear equations.

2. **Inequalities.** Put

$$H(x) = \int_0^x h(\xi)\, d\xi \quad \text{and} \quad K(x) = \int_0^x k(\xi)\, d\xi. \tag{2.1}$$

Let $x_1(t)$ be a solution of (1.1) subject to conditions (a)–(d).

We define the *total energy* as

$$V = \tfrac{1}{2}\dot{x}_1(t)^2 + H(x_1(t)). \tag{2.2}$$

If $x_1(t)$ is monotone on some interval then V becomes a function of x by replacing t in (2.2) by $x_1^{-1}(x)$. An easy calculation shows that if $|x_1(t)|<\varepsilon$, then V satisfies the differential inequality

$$|dV/dx| \leqq k(x)(2V - 2H)^{\alpha/2}. \tag{2.3}$$

LEMMA 1. *Let $V(x)$ be a positive solution of inequality* (2.3) *in the interval* $0<x<\varepsilon$ *and* $\lim_{x\to 0^+} V(x)=0$. *Then*

$$V(x) \leqq \tfrac{1}{2}(2-\alpha)^{2/(2-\alpha)}K(x)^{2/(2-\alpha)} \tag{2.4}$$

holds for all x, $0<x<\varepsilon$, *provided* $0<\alpha<2$. *If* $\alpha\geqq 2$, *then no solution of* (2.3) *exists satisfying* $\lim_{x\to 0^+} V(x)=0$.

PROOF. Suppose $0<\alpha<2$. From (2.3) we have $|V'|\leqq k(x)(2V)^{\alpha/2}$. Integrating, we have

$$\int_0^x V^{-\alpha/2}V'(\xi)\, d\xi = \frac{2}{2-\alpha} V(x)^{1-\alpha/2} \leqq 2^{\alpha/2}K(x)$$

and (2.4) follows.

If $\alpha\geqq 2$, the inequality

$$\int_0^x V^{-\alpha/2}V'(\xi)\, d\xi \leqq 2^{\alpha/2}K(x)$$

is a contradiction because the integral is an improper integral which diverges to $+\infty$.

From (2.4) we see that for α such that $0<\alpha<2$, the quantity $V(x)K(x)^{-2/(2-\alpha)}$ is bounded. The following lemma, which is the main result of this section gives a bound for the derivative of this quantity.

LEMMA 2. *Let* $0<\alpha<2$, *and let* $V(x)$ *be a monotone increasing positive solution of* (2.3) *defined on the interval* $(0, \varepsilon)$. *Put* $\beta=2/(2-\alpha)$. *Then, for* x *in* $(0, \varepsilon)$ *we have*

$$(d/dx)V(x)K(x)^{-\beta} \leqq \beta^{-\beta}\alpha^{\beta-1}K^{-1}k - \beta HK^{-\beta-1}k, \tag{2.5}$$

where equality holds only if

$$V(x) = \beta^{-\beta}\alpha^{\beta-1}K(x)^{\beta} + \text{const.} \tag{2.6}$$

PROOF. The differentiation on the left-hand side of (2.5) yields $-\beta K^{-\beta-1}kV+V'K^{-\beta}$. Using (2.3), this quantity is not greater than

$$\begin{aligned} &-\tfrac{1}{2}\beta kK^{-\beta-1}(k^{-2/\alpha}V'^{2/\alpha} + 2H) + K^{-\beta}V' \\ &\quad = -\tfrac{1}{2}kK^{-1}\beta^{-\beta+1}\alpha^{\beta}\{[(\alpha K/\beta)^{-\beta}k^{-2/\alpha}V'^{2/\alpha} \\ &\qquad\qquad - 2\alpha^{-\beta}(K/\beta)^{-\beta+1}k^{-1}V' + (2-\alpha)/\alpha] \\ &\qquad\qquad\qquad - (2-\alpha)/\alpha + 2H(x)(\alpha/\beta)^{-\beta}K^{-\beta}\}. \end{aligned} \tag{2.7}$$

Now the inequality

$$X^{2/\alpha} - 2X/\alpha + (2/\alpha - 1) \geqq 0 \tag{2.8}$$

holds provided $X>0$ and $0<\alpha<2$ with equality only if $X=1$. (See [**1**, p. 12].) We apply this inequality with $X=(\alpha K/\beta)^{-\beta+1}k^{-1}V'$ to find that the expression in (2.7) contained in square brackets is nonnegative. Inequality (2.5) follows, and the condition for equality (2.6) follows from the condition $X=1$ for equality in (2.8).

3. **An oscillation theorem.** In this section we show that the lemmas of the previous section can be used to give sufficient conditions for oscillation of solutions of (1.1).

We deal only with equations (1.1) for which the zero solution is globally asymptotically stable. Conditions for this type of stability for (1.1) have been the topic of a series of papers: [**2**], [**4**], and [**6**]. If our function $f(x, y)$ satisfies certain conditions at ∞ then [**6**, Theorem 3.5] asserts that all solutions of (1.1) are bounded; it easily follows from this that the zero solution is globally asymptotically stable. Since the behavior of $f(x, y)$ at ∞ is not specified by (a)–(d), it follows that for any $\varepsilon>0$, $k(x)>0$, and $\alpha>0$ there exist equations (1.1) satisfying (a)–(d) for which the zero solution is globally asymptotically stable. Hence the following theorem is not a vacuous assertion.

THEOREM 1. *Let the zero solution of* (1.1) *with assumptions* (a)–(d) *be globally asymptotically stable, and let* $\alpha\geqq 2$. *Then all solutions of* (1.1) *oscillate at* $+\infty$.

PROOF. Suppose $x(t)$ is a solution of (1.1), not identically zero, which does not oscillate at $+\infty$. One can easily show that the zeros of $x(t)$ and $\dot{x}(t)$ interlace each other. It follows from our assumption of global asymptotic stability that $\lim_{t\to\infty} x(t)=0$. Hence $x(t)$ is monotone for t sufficiently large, say $t \geqq t_0$. Thus $x(t)$ in the interval $t \geqq t_0$ generates a total energy function $V(x)>0$ which satisfies (2.3) and either $\lim_{x\to 0^+} V(x)=0$ or $\lim_{x\to 0^-} V(x)=0$. By Lemma 1, this is not possible for $\alpha \geqq 2$.

THEOREM 2. *Let the zero solution of* (1.1) *be globally asymptotically stable, and let assumptions* (a)–(d) *hold. Let* $0<\alpha<2$. *Let*

$$F(K(x), k(x), H(x), \alpha)$$

denote the right-hand side of inequality (2.5). *All solutions of* (1.1) *oscillate at* ∞ *provided*

$$\liminf_{B\to 0^+} \limsup_{A\to 0^+} \int_A^B F(K(x), k(x), H(x), \alpha)\, dx < 0 \tag{3.1}$$

and

$$\liminf_{B\to 0^+} \limsup_{A\to 0^+} \int_A^B F(-K(-x), k(-x), H(-x), \alpha)\, dx < 0. \tag{3.2}$$

Moreover, this assertion remains correct if (3.1) *is replaced by* (3.1′) *and/or* (3.2) *is replaced by* (3.2′) *as follows*:

$$\liminf_{B\to 0^+} \liminf_{A\to 0^+} \int_A^B F(K(x), k(x), H(x), \alpha)\, dx < -\tfrac{1}{2}(2-\alpha)^{2/(2-\alpha)}, \tag{3.1′}$$

$$\liminf_{B\to 0^+} \liminf_{A\to 0^+} \int_A^B F(-K(-x), k(-x), H(-x), \alpha)\, dx < -\tfrac{1}{2}(2-\alpha)^{2/(2-\alpha)}. \tag{3.2′}$$

PROOF. Suppose some solution $x(t)$ of (1.1) does not oscillate at ∞. As in the proof of Theorem 1 it follows that $\lim_{t\to\infty} x(t)=0$ and $x(t)$ must be monotone for t sufficiently large.

If $x(t)>0$ for t sufficiently large, we apply Lemma 2 to the corresponding total energy function $V(x)$. We find

$$\int_A^B F(K(x), k(x), H(x), \alpha)\, dx \geqq V(B)K(B)^{-2/(2-\alpha)} - V(A)K(A)^{-2/(2-\alpha)}. \tag{3.3}$$

From Lemma 1 we have

$$\begin{aligned} &\limsup_{x\to 0^+} V(x)K(x)^{-2/(2-\alpha)} \leqq \tfrac{1}{2}(2-\alpha)^{2/(2-\alpha)}, \\ &\liminf_{x\to 0^-} V(x)K(x)^{-2/(2-\alpha)} \geqq 0, \end{aligned} \tag{3.4}$$

and we find that neither (3.1) nor (3.1′) can hold.

In case $x(t)<0$ for large t, we observe that $-x(t)$ is a positive solution for the equation

$$\ddot{x} - f(-x, -\dot{x}) - h(-x) = 0. \tag{3.5}$$

If $V(x)$ is the total energy for (1.1) relative to the solution $x(t)$, then $V(-x)$ is the total energy for (3.5) relative to the solution $x(t)$. Arguing as above we find that neither (3.2) nor (3.2′) can hold.

4. **Sharpness of Theorem 2.** In this section we show that Theorem 2 is the best possible in the sense that the strict inequalities (3.1) and (3.2) cannot be replaced by weak inequalities. The examples which establish this sharpness are of interest because they show the qualitative differences which nonoscillatory solutions may exhibit depending on whether we have $0<\alpha<1$ or $1\leqq\alpha<2$. (We have already seen that nonoscillatory solutions do not exist in case $\alpha\geqq 2$.)

Put

$$A(\alpha) = \left|\frac{1-\alpha}{2-\alpha}\right|^{(2-\alpha)/(1-\alpha)} \left(\frac{(2-\alpha)\alpha}{2}\right)^{1/(1-\alpha)},$$

for $\alpha\neq 1$, $0<\alpha<2$, and put

$$\begin{aligned} g(t,\alpha) &= A(\alpha)(-t)^{(2-\alpha)/(1-\alpha)}, && 0<\alpha<1 \text{ and } t\leqq 0,\\ &= 0, && 0<\alpha<1 \text{ and } t>0,\\ &= A(\alpha)t^{(2-\alpha)/(1-\alpha)}, && 1<\alpha<2 \text{ and } t>0,\\ &= \exp(-t/2), && \alpha = 1. \end{aligned}$$

(Note that $g(t, \alpha)$ is not defined for $1<\alpha<2$ and $t\leqq 0$.)

For each α, $0<\alpha<2$, we see that $x(t)=g(t, \alpha)$ is a solution of (1.1) with

$$f(x, y) \equiv |y|^\alpha \operatorname{sgn} y, \quad h(x) \equiv ((2-\alpha)/2)^{2/(2-\alpha)}(\alpha x)^{\alpha/(2-\alpha)}.$$

A calculation shows that we have

$$V(x) = \alpha^{\alpha/(2-\alpha)}((2-\alpha)x/2)^{2/(2-\alpha)}$$

and that the integrands in (3.1) and (3.2) are identically zero. Moreover, the solutions $x(t)=f(t, \alpha)$ are nonoscillatory for each α, $0<\alpha<2$.

Note that in case $k(x)\equiv 1$ [**6**, Theorem 4.1] shows that the zero solution of (1.1) is globally asymptotically stable. Thus we have shown that the strict inequalities in (3.1) and (3.2) cannot be replaced by weak inequalities.

For $0<\alpha<1$, the solutions $x(t)=g(t, \alpha)$ are identically zero on an interval, namely $0\leqq t<\infty$, without being zero for all t. *This phenomenon is possible only if* α *is* <1. This fact is a special case of [**3**, p. 35, Exercise 6.4]. In fact, if $\alpha\geqq 1$, define

$$W(t) = \tfrac{1}{2}\dot{x}(t)^2 + H(x(t)),$$

where $x(t)$ is a solution of (1.1), subject to (a)–(d). Let I be any compact subinterval I of the domain of definition of $x(t)$ such that $|x(t)|<\varepsilon$ and $|\dot{x}(t)|<\varepsilon$ on I. Then there exists a constant M such that $(d/dt)W(t)\geqq -M(2W)^{(\alpha+1)/2}$ for all t in I. Since for $\alpha\geqq 1$ solutions of initial value problems for $du/dt=-M\,|2u|^{(\alpha+1)/2}$ are unique, and since $W(t)\geqq 0$, it follows from [**6**, p. 31, Theorem 6.1] that if for some τ in I one has $W(\tau)>0$ then $W(t)>0$ holds for all t in I such that $t>\tau$. This fact prevents a solution of (1.1) from vanishing on $0\leqq t<\infty$ without being identically zero.

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