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A NOTE ON K-COMMUTATIVITY OF MATRICES

EDMOND DALE DIXON AND NANCY WONG

ABSTRACT. It is the purpose of this paper to find in terms of parameters the most general matrix X which is K-commutative with respect to a given matrix A. The proofs will yield a method of rational construction for such a matrix X.

Introduction. In 1936 W. E. Roth [4] made an extensive investigation of K-commutativity of matrices. His theorems gave conditions for existence in terms of elementary divisors. It is the purpose of this paper to find in terms of parameters the most general matrix X which is K-commutative with respect to a given matrix A. The proofs will yield a method of rational construction for such a matrix X.

Let A and X be $n\times n$ matrices with elements from a field F. We denote $AX-XA$ by $[A, X]$ or $[A, X]_1$. Then $[A, X]_t=[A, [A, X]_{t-1}]$. The matrix X is said to be K-commutative with respect to A if $[A, X]_k=0$. It is easily seen from the above equations that we may assume that A has been reduced to rational canonical form. Furthermore, we may assume without loss of generality that the rational canonical form for A is $\operatorname{diag}\{C(f), C(g)\}$. In our theorems we find the blocks of X after it has been partitioned conformally with A. Thus we give conditions for the existence and give the construction of the most general matrix X_k satisfying $C(f)X_k=X_kC(g)+X_{k-1}$, $C(f)X_{k-1}=X_{k-1}C(g)+X_{k-2}, \cdots, C(f)X_1=X_1C(g)$ where $C(f)$ and $C(g)$ are companion matrices of the polynomials $f(x)$ and $g(x)$ and where $f(x)$ divides $g(x)$ or $g(x)$ divides $f(x)$.

The methods used were developed by W. V. Parker [1] in his investigation of the equation $AX=XB$.

Two-commutative case. We now study the equation

$$C(f)Y = YC(g) + X, \quad \text{where } C(f)X = XC(g). \tag{1}$$

THEOREM 1. *If $C(f)$ and $C(g)$ are companion matrices of the polynomials $f(x)=x^m-\sum_{i=1}^m a_ix^{i-1}$ and $g(x)$ of degrees m and n respectively over the field F, then the $m\times n$ matrices Y and X, over F, satisfy the equation*

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$C(f)Y=YC(g)+X$, where $C(f)X=XC(g)$, if and only if the rows of X and Y are U, $UC(g)$, $\cdots$, $UC^{m-1}(g)$ and V, $VC(g)+U$, $\cdots$, $VC^{m-1}(g)+(m-1)UC^{m-2}(g)$ respectively, where U and V are $1\times n$ matrices satisfying the following conditions:

$$Vf[C(g)] + Uf'[C(g)] = 0 \quad \text{and} \quad Uf[C(g)] = 0. \tag{2}$$

PROOF. Parker [1] has shown that the rows of X are U, $UC(g)$, $\cdots$, $UC^{m-1}(g)$ where $Uf[C(g)]=0$. If the rows of Y are V_1, V_2, $\cdots$, V_m, then the rows of $C(f)Y$ are V_2, V_3, $\cdots$, V_m and $\sum_{i=1}^m a_iV_i$. The rows of $YC(g)+X$ are $V_1C(g)+U$, $V_2C(g)+UC(g)$, $\cdots$, $V_mC(g)+UC^{m-1}(g)$. Setting the corresponding rows of (1) equal gives

$$\begin{aligned} V_2 &= V_1C(g) + U, \\ V_3 &= V_2C(g) + UC(g) = V_1C^2(g) + 2UC(g), \\ &\cdot \\ &\cdot \\ &\cdot \\ V_m &= V_{m-1}C(g) + UC^{m-2}(g) = V_1C^{m-1}(g) + (m-1)UC^{m-2}(g), \\ \sum_{i=1}^m a_iV_i &= V_mC(g) + UC^{m-1}(g) = V_1C^m(g) + mUC^{m-1}(g), \end{aligned} \tag{3}$$

then

$$\begin{aligned} a_1V_1 + a_2V_2 + \cdots + a_mV_m &= a_1V_1 + a_2[V_1C(g) + U] + \cdots \\ &\quad + a_m[V_1C^{m-1}(g) + (m-1)UC^{m-2}(g)] \\ &= V_1C(g) + mUC^{m-1}(g). \end{aligned}$$

Transposing and collecting the coefficients of V_1 and the coefficients of U gives

$$\begin{aligned} &V_1[C^m(g) - (a_1I + a_2C(g) + \cdots + a_mC^{m-1}(g))] \\ &\quad + U[mC^{m-1}(g) - (a_2I + 2a_3C(g) + \cdots + (m-1)a_mC^{m-2}(g))] = 0. \end{aligned}$$

That is $V_1f[C(g)]+Uf'[C(g)]=0$. Since all the steps of this proof are reversible, the converse holds.

It should be noted that if $g(x)$ divides $f(x)$, then $V_1f[C(g)]=0$ and $Uf[C(g)]=0$ and V_1 is arbitrary. If $f(x)$ divides $g(x)$, a $1\times n$ matrix different from zero exists satisfying (3).

Equations (2) are satisfied if and only if

$$V[C(g)]f[C(g)] + U[C(g)]f'[C(g)] = 0$$

and

$$U[C(g)]f[C(g)] = 0.$$

Thus V and U must be such that

$$(4) \quad V(x)f(x) + U(x)f'(x) = g(x)q(x) \quad \text{and} \quad U(x)f(x) = g(x)k(x)$$

where $k(x)$ and $q(x)$ are arbitrary polynomials such that the degrees of $g(x)k(x)$ and $g(x)q(x)$ are not greater than $m+n-1$. Therefore the $1\times n$ matrices V and U are obtained through their associated polynomials.

K-commutative case. We now study the system

$$(5) \quad \begin{aligned} C(f)X_k &= X_kC(g) + X_{k-1}, \\ C(f)X_{k-1} &= X_{k-1}C(g) + X_{k-2}, \\ &\cdot \\ &\cdot \\ &\cdot \\ C(f)X_2 &= X_2C(g) + X_1, \end{aligned}$$

where $C(f)X_1=X_1C(g)$ and where $C(f)$ and $C(g)$ are companion matrices of the polynomials $f(x)=x^m-\sum_{i=1}^m a_ix^{i-1}$ and $g(x)$ of degrees m and n respectively.

THEOREM 2. *Let $C(f)$ and $C(g)$ be as given in* (5). *Then the set of $m\times n$ matrices $\{X_1, X_2, \cdots, X_k\}$, over F, satisfy* (5) *if and only if the rows of $X_1, X_2, \cdots, X_k$ are*

$$\binom{0}{0}U_1,\ \binom{1}{0}U_1C(g), \cdots, \binom{m-1}{0}U_1C^{m-1}(g)$$

$$\binom{0}{0}U_2,\ \binom{1}{0}U_2C(g) + \binom{1}{1}U_1, \cdots, \binom{m-1}{0}U_2C^{m-1}(g) + \binom{m-1}{1}U_1C^{m-2}(g)$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$\binom{0}{0}U_k,\ \binom{1}{0}U_kC(g) + \binom{1}{1}U_{k-1}, \cdots, \binom{m-1}{0}U_kC^{m-1}(g) + \binom{m-1}{1}U_{k-1}C^{m-2}(g) + \cdots + \binom{m-1}{k-1}U_1C^{m-k}(g)$$

successively, where $U_1, U_2, \cdots, U_{k-1}$, and U_k are a set of $1\times n$ matrices *satisfying the following equations*:

$$
(6)\qquad
\begin{aligned}
&U_k f[C(g)] + U_{k-1} f'[C(g)] + \cdots + U_1 \frac{f^{(k-1)}[C(g)]}{(k-1)!} = 0,\\
&U_{k-1} f[C(g)] + U_{k-2} f'[C(g)] + \cdots + U_1 \frac{f^{(k-2)}[C(g)]}{(k-2)!} = 0,\\
&\quad\vdots\\
&U_1 f[C(g)] = 0.
\end{aligned}
$$

PROOF. We have shown that if $K=2$ then the rows of X_2 are $\binom{0}{0}U_2$, $\binom{1}{0}U_2C(g)+\binom{1}{1}U_1, \ldots, \binom{m-1}{0}U_2C^{m-1}(g)+\binom{m-1}{1}U_1C^{m-2}(g)$ where

$$U_2 f[C(g)] + U_1 f'[C(g)] = 0 \quad \text{and} \quad Uf[C(g)] = 0.$$

We now proceed by induction to construct the rows of X_k for arbitrary K.

Assume that if $K=p-1$ the rows of X_{p-1} are

$$\binom{0}{0}U_{p-1}, \binom{1}{0}U_{p-1}C(g) + \binom{1}{1}U_{p-2}, \cdots, \binom{m-1}{0}U_{p-1}C^{m-1}(g) + \binom{m-1}{1}U_{p-2}C^{m-2}(g) + \cdots + \binom{m-1}{p-2}U_1C^{m-p+1}(g)$$

where

$$
(7)\qquad
\begin{aligned}
&U_{p-1} f[C(g)] + U_{p-2} f'[C(g)] + \cdots + U_1 \frac{f^{(p-2)}[C(g)]}{(p-2)!} = 0,\\
&U_{p-2} f[C(g)] + U_{p-3} f'[C(g)] + \cdots + U_1 \frac{f^{(p-3)}[C(g)]}{(p-3)!} = 0,\\
&\quad\vdots\\
&U_1 f[C(g)] = 0.
\end{aligned}
$$

Now set $C(f)X_p = X_pC(g) + X_{p-1}$. If the rows of X_p are $V_1 = U_p$, $V_2, \cdots, V_m$, then the rows of the left side of (7) are $V_2, V_3, \cdots, V_m$ and $\sum_{i=1}^{m} a_i V_i$. The rows of the right side of (7) are

$$V_1C(g) + \binom{0}{0}U_{p-1} = U_pC(g) + \binom{0}{0}U_{p-1},$$

$$V_2C(g) + \binom{1}{0}U_{p-1}C(g) + \binom{1}{1}U_{p-2}, \cdots,$$

$$V_mC(g) + \binom{m-1}{0}U_{p-1}C^{m-1}(g) + \cdots + \binom{m-1}{p-2}U_1C^{m-p+1}(g).$$

According to Theorem 1, the rows of X_p are

$$\binom{0}{0}U_p,\ \binom{1}{0}U_pC(g)+\binom{1}{1}U_{p-1},\cdots,\binom{m-1}{0}U_pC^{m-1}(g)$$
$$+\binom{m-1}{1}U_{p-1}C^{m-2}(g)+\cdots+\binom{m-1}{p-1}U_1C^{m-p}(g)$$

and

$$\sum_{i=1}^{m} a_iV_i=\binom{m}{0}U_pC^m(g)+\binom{m}{1}U_{p-1}C^{m-1}(g)+\cdots$$
$$+\binom{m}{p-1}U_1C^{m-p+1}(g)$$

by setting the corresponding rows of both sides of (7) equal. Replacing V_i by $V_i=\sum_{j=0}^{i-1}\binom{i-1}{0}U_{p-j}C^{i-j-1}(g)$, $i=2,3,\cdots,m$, where $U_{p-j}=0$ if $j>p$, we have

$$\sum_{i=1}^{m} a_iV_i=a_1U_p+a_2\left[U_pC(g)+\binom{1}{1}U_{p-1}\right]+\cdots$$
$$+a_m\left[\binom{m-1}{0}U_pC^{m-1}(g)+\binom{m-1}{1}U_{p-1}C^{m-2}(g)+\cdots\right.$$
$$\left.+\binom{m-1}{p-1}U_1C^{m-p}(g)\right]$$
$$=\binom{m}{0}U_pC^m(g)+\binom{m}{1}U_{p-1}C^{m-1}(g)+\cdots$$
$$+\binom{m}{p-1}U_1C^{m-p+1}(g).$$

Transposing and collecting the coefficients of $U_p, U_{p-1},\cdots,U_1$ we get

$$U_pf[C(g)]+U_{p-1}f'[C(g)]+\cdots+U_1\frac{f^{(p-1)}[C(g)]}{(p-1)!}=0.$$

Since all the steps of this proof are reversible, the converse holds.

It should be noted that if $g(x)$ divides $f(x)$, then

$$U_pf[C(g)]=0,$$
$$U_{p-1}f[C(g)]=0,$$
$$\vdots$$
$$U_1f[C(g)]=0$$

and U_p is arbitrary.

Equations (6) are satisfied if and only if

$$U_k[C(g)]f[C(g)] + U_{k-1}[C(g)]f'[C(g)] + \cdots + U_1[C(g)]\frac{f^{(k-1)}[C(g)]}{(k-1)!} = 0,$$

$$U_{k-1}[C(g)]f[C(g)] + U_{k-2}[C(g)]f'[C(g)] + \cdots + U_1[C(g)]\frac{f^{(k-2)}[C(g)]}{(k-2)!} = 0,$$

$$\vdots$$

$$U_1[C(g)]f[C(g)] = 0,$$

thus $U_1, U_2, \cdots, U_{k-1}$ must be such that

$$U_1(x)f(x) = g(x)k_1(x),$$

$$U_2(x)f(x) + U_1(x)f'(x) = g(x)k_2(x),$$

$$\vdots$$

$$U_{k-1}(x)f(x) + U_{k-2}(x)f'(x) + \cdots + U_1(x)\frac{f^{(k-2)}(x)}{(k-2)!} = g(x)k_{p-1}(x),$$

where $k_i(x)$ for $i=1, 2, \cdots, p-1$ are arbitrary polynomials such that the degree of $g(x)k_i(x)$ is not greater than $m+n-2$. Therefore the set of $1\times n$ matrices $U_1, U_2, \cdots, U_{k-1}$ are obtained through their associated polynomials.

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Department of Mathematics, Tennessee Technological University, Cookeville, Tennessee 38501