A FIXED POINT THEOREM FOR $(n - 2)$ -CONNECTED n -POLYHEDRA

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ABSTRACT. The main result of this paper is that, for $n \geq 4$, a finite $(n-2)$ -connected polyhedron K of dimension n admits a fixed point free map if either $H_{n-1}(K; Q)$ or $H_n(K; Q)$ is nonzero, where Q is the field of rational numbers. This result is obtained by first retracting K onto a subpolyhedron C of dimension n or $n-1$ which has no local separating points. It is then shown that C admits a map with Lefschetz number zero, and it follows from a theorem of Shi that C does not have the fixed point property. The proof involved may also be applied when K is a 3-dimensional simply connected polyhedron and the subpolyhedron C is also of dimension 3.

1. Introduction. The main purpose of this paper is to show that, for $n \geq 4$, a finite $(n-2)$ -connected polyhedron K of dimension n admits a fixed point free map if either $H_{n-1}(K; Q)$ or $H_n(K; Q)$ is nonzero. This result may be regarded as a higher dimensional analogue of the well-known fact that if $H_1(K; Q) \neq 0$, then the space K admits a fixed point free map (obtained by retracting K onto a simple closed curve). The proof of the present theorem is obtained by retracting K onto a subpolyhedron C of dimension n or $n-1$ which has no local separating points, and then constructing a map of C which has Lefschetz number 0. It follows from a theorem of Shi [3] that C fails to have the fixed point property. The reader may use Fadell [1] as a general reference.

A map is a continuous function. If $f: X \rightarrow Y$ is a map, then $M(f)$ will denote the mapping cylinder of f . Unless otherwise indicated, $H(X)$ will denote the singular homology of X with integer coefficients, while $\bar{H}(X; Q)$ denotes the reduced singular homology of X with rational coefficients. If c is a cocycle of X , then $[c]$ will be the image of c in the cohomology group of X . A bouquet of m -spheres is a CW complex consisting of a base point x_0 and a collection of m -cells $\{S_\alpha\}$ whose boundaries are attached to x_0 . A polyhedron K satisfies the Shi condition if the dimension of K is at least 3 and K has no local separating points. The j -sphere is S^j . The term "fixed point property" will be abbreviated "f.p.p."

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2. The Theorem. The following lemma gives a sufficient condition for retracting a CW complex of dimension $m+1 \geq 3$ onto an m -dimensional subcomplex which is a bouquet of spheres.

LEMMA. *Let K be a CW complex of dimension $m+1 \geq 3$, let B be an m -dimensional bouquet of spheres which is a subcomplex of K , and let $i: B \subset K$. If $i_*: H_m(B) \rightarrow H_m(K)$ has a left inverse h , then B is a retract of K .*

PROOF. Cellular homology is used for this proof, and the obstruction theory involved is found in Hu [2]. Consider the following commutative diagram.

$$\begin{array}{ccccc}
 & & \pi_m(B, x_0) & & \\
 & & \uparrow a & & \\
 C_m(B) & \xrightarrow{i} & C_m(K) & \xrightarrow{j} & H_m(K^m; K^{m-1}) \\
 \uparrow \theta & & & & \uparrow \varphi \downarrow \psi \\
 H_m(B) & \xrightleftharpoons[h]{i_*} & H_m(K) & \xleftarrow[\eta]{} & H_m(K^m)
 \end{array}$$

$C_m(B)$ and $C_m(K)$ are groups of cellular chains, a is a cocycle in $Z^m(B; \pi_m(B, x_0))$, K^m and K^{m-1} are skeleta of K , θ and j are isomorphisms, η and ψ are induced by inclusion, and $\varphi\psi$ is the identity. Then $a\theta h\eta\varphi j$ is an m -cocycle for K . Now $i^*[a\theta h\eta\varphi j] = [a\theta h\eta\varphi j i] = [a]$, so $i^*: H^m(K; \pi_m(B, x_0)) \rightarrow H^m(B; \pi_m(B, x_0))$ is an epimorphism. Thus $\delta: H^m(B; \pi_m(B, x_0)) \rightarrow H^{m+1}(K, B; \pi_m(B, x_0))$ is zero. But the image of δ contains the obstruction to extending a map $B \rightarrow B$ to a map $K \rightarrow B$. Since any such map may be extended, B is a retract of K .

THEOREM. *If K is a finite $(n-2)$ -connected polyhedron of dimension $n \geq 4$ and $\tilde{H}(K; Q) \neq 0$, then K does not have the f.p.p.*

PROOF. Let us first suppose that K satisfies the Shi condition. For either $j=n$ or $j=n-1$, $H_j(K; Q) \neq 0$. Since the Hurewicz homomorphism $\pi_j(K) \rightarrow H_j(K)$ is onto, there is a cellular map $f: S^j \rightarrow K$ such that $f_*: H_j(S^j) \rightarrow H_j(K)$ is a monomorphism and the image of f_* is a direct factor of $H_j(K)$. This implies that f_* has a left inverse. Thus if $i: S^j \subset M(f)$, i_* also has a left inverse. Applying the lemma, S^j is a retract of $M(f)$ and hence $M(f)$ admits a fixed point free map. Thus K admits a map whose Lefschetz number is 0, and from results in [3] it follows that K has a fixed point free map.

For the general case, let $v_1, \dots, v_q$ be the vertices of K , and let $C_1, \dots, C_p$ be the closures of the components of $K \setminus \{v_1, \dots, v_q\}$. Then there is an

$(n-2)$ -connected C_k which satisfies the Shi condition, for which $\tilde{H}(C_k; Q) \neq 0$. Since C_k is a retract of K which admits a fixed point free map, K does also.

REMARK. The proof of the theorem may also be applied if the dimension of K is 3 and there is some C_k of dimension 3 with $\tilde{H}(C_k; Q) \neq 0$. For example, if the dimension of K is 3, K is simply connected, and $H_3(K; Q) \neq 0$, then K fails to have the f.p.p.

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