GRONWALL'S INEQUALITY FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN TWO INDEPENDENT VARIABLES

DONALD R. SNOW

ABSTRACT. This paper presents a generalization for systems of partial differential equations of Gronwall's classical integral inequality for ordinary differential equations. The proof is by reducing the vector integral inequality to a vector partial differential inequality and then using a vector generalization of Riemann's method to obtain the final inequality. The final inequality involves a matrix function in the integrand which is a generalization of the scalar Riemann function. The proof includes a successive approximations argument to guarantee the existence and positivity property of this matrix function. The inequality is applied to prove a uniqueness theorem for a nonlinear vector hyperbolic partial differential equation, a comparison theorem for a linear hyperbolic vector partial differential equation, and a continuous dependence theorem for a nonlinear vector boundary value problem. The inequality also appears to have many applications in stability problems and in numerical solutions of partial differential equations. All of these results hold for the corresponding Volterra integral equations and the method of proof of the main result shows that the function on the right-hand side of the final inequality is the solution of the integral equation and hence is the maximal solution of the original inequality.

1. Introduction. [1] gave a generalization of Gronwall's classical one independent variable inequality [2] (also called Bellman's Lemma [3]) to a scalar integral inequality in two independent variables and applied the result to three problems in partial differential equations.¹ The present paper extends these known results in the two independent variable case to a

Received by the editors November 16, 1970.

AMS 1970 subject classifications. Primary 35A05, 35B05, 35B30, 35B35, 35B45, 35L15, 35L20, 35L55, 35L60; Secondary 34A40, 34C10, 35A35, 35A40, 35C15, 45D05, 45F05, 45M10.

Key words and phrases. Gronwall's inequality, differential inequality, integral inequality, hyperbolic systems, system of Volterra integral equations, uniqueness theorems, comparison theorems, continuous dependence, stability, numerical solution, vector characteristic initial value problems, Riemann's method, Riemann functions, maximal solutions, matrix Lipschitz condition.

¹ Reference [4] contains much information on the scalar generalizations of Gronwall's inequality including a generalization to several independent variables.

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system of integral inequalities and applies the result to vector partial differential equations. As discussed in [1] it appears that these inequalities will have as many applications for partial differential equations as the classical Gronwall inequality has had for ordinary differential equations. The areas of applications are uniqueness theorems, comparison theorems, continuous dependence results, stability, and numerical computations.

The main result is obtained by reducing the vector integral inequality to a vector differential inequality and then integrating it by generalizing Riemann's method to apply to vector hyperbolic PDE's. As in [1], the method of proof shows that the right-hand side of the final inequality is the solution of the corresponding vector Volterra integral equation and hence is the maximal solution of the original inequality. This method can be used to solve the corresponding type of vector Volterra integral equations.

§2 contains the main theorem, its proof and corollaries, and §3 gives some applications.

2. Main theorem and corollaries.

THEOREM. Suppose $\phi(x, y)$ and $a(x, y)$ are continuous n -vector functions on a domain D and $B(x, y)$ is a continuous symmetric, nonnegative (i.e. $b_{ij} \geq 0$, all i and j), $n \times n$ matrix function on D . Let $P_0(x_0, y_0)$ and $P(x, y)$ be two points in D such that $(x - x_0)(y - y_0) \geq 0$ and let R be the rectangular region whose diagonal is the line joining the points P_0 and P . Let $V(s, t; x, y)$ be the $n \times n$ matrix function satisfying the matrix characteristic initial value problem

$$L[V] = V_{st} - B(s, t)V = 0, \quad V(s, y) = V(x, t) = I,$$

where I is the identity matrix. Let D^+ be the connected subdomain of D

FIGURE 1. Diagram showing the regions involved

containing P and on which $V(s, t; x, y)$ is nonnegative. (See Figure 1 and the Lemma.)

Then, if $R \subset D^+$ and ϕ satisfies

$$(1) \quad \phi(x, y) \leq a(x, y) + \int_{x_0}^x \int_{y_0}^y B(s, t) \phi(s, t) ds dt$$

where the inequality holds componentwise, ϕ also satisfies

$$(2) \quad \phi(x, y) \leq a(x, y) + \int_{x_0}^x \int_{y_0}^y V^T(s, t; x, y) B(s, t) a(s, t) ds dt.$$

The function $V(s, t; x, y)$ is a matrix generalization of a Riemann function relative to the point $P(x, y)$ for the selfadjoint operator L . There is such a function and region D^+ on which $V \geq 0$ as proven in the following Lemma.

LEMMA. Let $B(s, t)$ be a continuous matrix function. Then the matrix characteristic initial value problem

$$(3) \quad L[V] = V_{st} - B(s, t)V = 0,$$

$$(4) \quad V(s, y) = V(x, t) = I,$$

where I is the identity matrix, has a unique solution $V(s, t; x, y)$ for s and t near to $P(x, y)$ and satisfying $(s-x)(t-y) \geq 0$. This solution is continuous and if $B(s, t)$ is nonnegative, so is $V(s, t; x, y)$.

PROOF. The proof is by a successive approximations argument. It is easily seen that the integral equation

$$(5) \quad V(s, t) = I + \int_x^s \int_y^t B(\xi, \eta) V(\xi, \eta) d\xi d\eta$$

is equivalent to equation (3) with conditions (4).

Let T represent the transformation

$$TV = \int_x^s \int_y^t B(\xi, \eta) V(\xi, \eta) d\xi d\eta,$$

so equation (5) is $V = I + TV$.

Let $V_0(s, t) \equiv I$ and define $V_{n+1} = I + TV_n$. Since TV is continuous if V is, it follows immediately by induction that V_n is defined and continuous for all n . Let $\|\cdot\|$ be any matrix norm. Then since $(s-x)(t-y) \geq 0$,

$$(6) \quad \begin{aligned} \|TV\| &= \left\| \int_x^s \int_y^t B(\xi, \eta) V(\xi, \eta) d\xi d\eta \right\| \\ &\leq \int_x^s \int_y^t \|B(\xi, \eta)\| \cdot \|V(\xi, \eta)\| d\xi d\eta \leq \frac{1}{2} \max \|V(\xi, \eta)\| \end{aligned}$$

if we restrict s and t to be close enough to (x, y) .

Then

$$\begin{aligned}\|V_{k+1} - V_k\| &= \|T(V_k - V_{k-1})\| \leq \frac{1}{2} \max \|V_k - V_{k-1}\| \\ &\leq \cdots \leq 2^{-k} \max \|V_1 - V_0\|.\end{aligned}$$

Since $V_{n+1} = V_0 + \sum_{k=0}^n (V_{k+1} - V_k)$, V_{n+1} is the n th partial sum of a matrix series dominated in norm by a convergent geometric series, namely $\max \|V_1 - V_0\| \sum_{k=0}^{\infty} 2^{-k}$.

Therefore the matrix sequence $\{V_n\}$ converges uniformly on the domain where (6) holds. Since each V_n is continuous, the limit function $V(s, t; x, y)$ is also. To see that V is a solution to (5), note that $I + TV = I + T(\lim V_n) = I + \lim TV_n = \lim(I + TV_n) = \lim V_{n+1} = V$. To see that V is unique, suppose W is also a solution. Then $V - W = T(V - W)$ so

$$\|V - W\| = \|T(V - W)\| \leq \frac{1}{2} \max \|V - W\|$$

which is possible only if $\|V - W\| = 0$; i.e. $V \equiv W$.

Now suppose $B(s, t) \geq 0$. Then, if $V \geq 0$, $TV \geq 0$, since $(s-x)(t-y) \geq 0$. Since $V_0 \equiv I \geq 0$, it follows by induction that $V_n \geq 0$ for all n . But then the limit function satisfies $V(s, t; x, y) \geq 0$ also. ■

PROOF OF THEOREM. Let

$$(7) \quad u(x, y) = \int_{x_0}^x \int_{y_0}^y B(s, t) \phi(s, t) ds dt.$$

Then $u_{xy} = B(x, y)\phi$ and since $B \geq 0$ and (1) holds,

$$(8) \quad u_{xy} = B\phi \leq B(a + u) \quad \text{or} \quad L[u] = u_{xy} - Bu \leq Ba.$$

This is a hyperbolic vector partial differential inequality for u . The initial conditions for u are

$$(9) \quad u(x_0, y) = u(x, y_0) = 0.$$

The operator L is selfadjoint. We note that for any $u, v \in C^2$,

$$v^T L[u] - u^T L[v] = v^T u_{xy} - v^T Bu - u^T v_{xy} + u^T Bv.$$

The terms here are all scalars and since B is symmetric, the second and fourth terms on the right cancel. The right-hand side is

$$\begin{aligned}(10) \quad &= -(u^T v_y)_x + u_x^T v_y + (v^T u_x)_y - v_y^T u_x \\ &= -(u^T v_y)_x + (v^T u_x)_y.\end{aligned}$$

For P_0 and P as required in the hypothesis we label the directed sides and corners of the rectangle R as shown in Figure 2.

FIGURE 2. Directed path around R

Using s and t as the independent variables in identity (10), integrating it over R , and using Green's Theorem, we get

$$\begin{aligned}
 \iint_R \{v^T L[u] - u^T L[v]\} ds dt &= \iint_R \{-(u^T v_t)_s + (v^T u_s)_t\} ds dt \\
 (11) \qquad \qquad \qquad &= - \int_C v^T u_s ds + u^T v_t dt \\
 &= - \int_{C_2+C_4} v^T u_s ds - \int_{C_1+C_3} u^T v_t dt.
 \end{aligned}$$

This holds for any functions in C^2 .

For any $u \in C^2$ which also satisfies the initial conditions (9), $u=0$ on C_3 and $u=u_s=0$ on C_4 . Thus (11) reduces to

$$(12) \qquad \qquad \qquad - \int_{C_2} v^T u_s ds - \int_{C_1} u^T v_t dt.$$

Now suppose the vectors $v^i(s, t; x, y)$ are the columns of the matrix $V(s, t; x, y)$ of the Lemma. Then $L[v^i]=0$ and $v^i(s, y)=v^i(x, t)=e^i$, the i th column of the identity matrix. Thus $v_t^i=0$ on C_1 so (12) reduces to $\iint_R v^{iT} L[u] ds dt = - \int_{C_2} e^{iT} u_s ds = u_i(P)$ where the subscript refers to the component of the vector. Using the matrix V this becomes

$$u(P) = \iint_R V^T(s, t; x, y) L[u] ds dt.$$

For u defined by (7), since (8) holds and since $V \geq 0$ by the Lemma, we get

$$(13) \quad u(P) \leq \iint_R V^T B a \, ds \, dt.$$

This gives an upper bound for the integral term in (1) so that (2) results. ■

The matrix V is a generalization of a scalar Riemann function and when ϕ , a , and B are scalars, it reduces to the Riemann function relative to the point $P(x, y)$ for the operator L .

We note that by the method of proof, if equality held in (1), there would have been equality in (8) and (13) regardless of the nonnegativity of V , so the right-hand side of (2) is the solution to the Volterra integral equation corresponding to (1). Since the right-hand side of (2) is a solution of the inequality (1) and is an upper bound for all such solutions, it is the *maximal solution* of (1).

COROLLARY 1. *If $\phi(x, y) \leq \int_{x_0}^x \int_{y_0}^y B(s, t) \phi(s, t) \, ds \, dt$, $B \geq 0$, and $(x - x_0) \cdot (y - y_0) \geq 0$, then $\phi(x, y) \leq 0$.*

PROOF. Let a of the theorem be the zero vector. ■

COROLLARY 2. *If a is a constant vector then $\phi(x, y) \leq V^T(x_0, y_0; x, y)a$ or by the symmetry of V in its variables, $\phi(x, y) \leq V^T(x, y; x_0, y_0)a$.*

PROOF. For a constant vector a , (2) becomes

$$\phi \leq \left\{ I + \int_{x_0}^x \int_{y_0}^y V^T(s, t; x, y) B(s, t) \, ds \, dt \right\} a.$$

By (3) and the symmetry of B the integrand is $V_{st}^T(s, t; x_0, y_0)$. Integrating and using conditions (4) which V satisfies we get the desired result. ■

COROLLARY 3. *If inequality (1) is reversed, then so is inequality (2).*

PROOF. Since $B \geq 0$, we still have $V \geq 0$. Hence reversing inequality (1) reverses (8) and (13), so (2) is reversed. ■

3. Applications. The three applications considered here are the vector analogs of those in [1]. They are not stated as theorems so as not to obscure the main ideas with technical details.

EXAMPLE 1. Consider the nonlinear vector hyperbolic partial differential equation

$$(14) \quad u_{xy} = f(x, y, u)$$

with boundary conditions prescribed on $x = x_0$ and $y = y_0$. We assume f

is continuous and satisfies a *matrix Lipschitz condition*; i.e. there is a non-negative matrix K such that

$$|f(x, y, u) - f(x, y, \bar{u})| \leq K |u - \bar{u}|$$

where the absolute values are taken componentwise and u and $\bar{u}$ are any vectors. If each component of f satisfies a scalar Lipschitz condition, then K reduces to a diagonal matrix and letting k be the largest of the diagonal elements we could use kI , where I is the identity matrix, for the Lipschitz constant matrix.

Using the boundary conditions the equivalent vector Volterra integral equation for (14) is

$$u(x, y) = g(x, y) + \int_{x_0}^x \int_{y_0}^y f(s, t, u(s, t)) ds dt,$$

where g depends on the boundary values. Then for any two solutions u and $\bar{u}$ of the integral equation we have

$$u - \bar{u} = \int_{x_0}^x \int_{y_0}^y [f(s, t, u) - f(s, t, \bar{u})] ds dt.$$

Hence, if $(x-x_0)(y-y_0) \geq 0$ and we take absolute values componentwise we get

$$|u - \bar{u}| \leq \int_{x_0}^x \int_{y_0}^y |f(s, t, u) - f(s, t, \bar{u})| ds dt \leq \int_{x_0}^x \int_{y_0}^y K |u - \bar{u}|.$$

By Corollary 1, $|u - \bar{u}| \leq 0$ componentwise, so $u \equiv \bar{u}$; i.e., the solution is unique.

This result can also be proven by using a norm instead of absolute values componentwise to get

$$\|u - \bar{u}\| \leq \int_{x_0}^x \int_{y_0}^y \|f(s, t, u) - f(s, t, \bar{u})\| ds dt,$$

and noting that the matrix Lipschitz condition implies a norm Lipschitz condition so that

$$\|u - \bar{u}\| \leq \int_{x_0}^x \int_{y_0}^y \tilde{K} \|u - \bar{u}\| ds dt \quad \text{where } \tilde{K} \text{ is a scalar.}$$

The desired conclusion now follows from the scalar two independent variable Gronwall inequality presented in [1].

The first method, the vector approach, may be easier to verify for some problems.

EXAMPLE 2. As a second example consider the vector characteristic initial value problem for the linear hyperbolic partial differential equation

$$u_{xy} - B(x, y)u = f(x, y),$$

where B is a nonnegative continuous matrix and the vector u is prescribed on $x=x_0$ and $y=y_0$. Using the boundary conditions, this problem is equivalent to the vector Volterra integral equation

$$u(x, y) = F(x, y) + \int_x^{x_0} \int_{y_0}^y B(s, t) u(s, t) ds dt$$

where F is computed from f and the boundary conditions and we assume F is continuous. Now suppose the vectors v and w satisfy

$$v \leq F + \int_{x_0}^x \int_{y_0}^y Bv ds dt, \quad w \geq F + \int_{x_0}^x \int_{y_0}^y Bw ds dt.$$

Then the Theorem and Corollary 3 show that for any solution u to the boundary value problem, we have $v \leq u \leq w$. This is a componentwise comparison theorem for the solution u .

EXAMPLE 3. Consider the two vector boundary value problems:

$$\begin{aligned} u_{xy} &= f(x, y, u), & U_{xy} &= F(x, y, U), \\ u(x_0, y) &= g(y), & U(x_0, y) &= G(y), \\ u(x, y_0) &= h(x), & U(x, y_0) &= H(x), \\ g(y_0) &= h(x_0), & G(y_0) &= H(x_0), \end{aligned}$$

where all the functions are continuous and f satisfies a matrix Lipschitz condition. This, of course, implies a norm Lipschitz condition as in Example 1 so a norm type continuous dependence result can be obtained by using the scalar inequality in [1]. Here we will obtain a componentwise result. We write the equivalent vector integral equations and subtract to get

$$\begin{aligned} u - U &= (g - G) + (h - H) - [g(y_0) - G(y_0)] \\ &\quad + \int_{x_0}^x \int_{y_0}^y [f(s, t, u) - F(s, t, U)] ds dt. \end{aligned}$$

By adding and subtracting $f(s, t, U)$ in the integrand and taking absolute values componentwise, we get, for $(x-x_0)(y-y_0) \geq 0$,

$$\begin{aligned} |u - U| &\leq |g - G| + |h - H| + |g(y_0) - G(y_0)| \\ &\quad + \int_{x_0}^x \int_{y_0}^y |f(s, t, u) - f(s, t, U)| ds dt \\ &\quad + \int_{x_0}^x \int_{y_0}^y |f(s, t, U) - F(s, t, U)| ds dt. \end{aligned}$$

If $|g - G| \leq \varepsilon$, $|h - H| \leq \varepsilon$, and $|f(s, t, U) - F(s, t, U)| \leq \varepsilon$ where ε is a

nonnegative vector, then

$$|u - U| \leq 3\varepsilon + \varepsilon(x - x_0)(y - y_0) + \int_{x_0}^x \int_{y_0}^y K |u - U| \, ds \, dt$$

where K is the nonnegative Lipschitz constant matrix for f . By the Theorem, we have

$$\begin{aligned} |u - U| &\leq 3\varepsilon + \varepsilon(x - x_0)(y - y_0) \\ &\quad + \int_{x_0}^x \int_{y_0}^y V^T(s, t; x, y) K [3\varepsilon + \varepsilon(s - x_0)(t - y_0)] \, ds \, dt \\ &= M(x, y)\varepsilon \end{aligned}$$

where M is a continuous matrix function which can be computed at least in theory. On a compact domain in the xy -plane, M is bounded so the solution to the boundary value problem we started with depends continuously on f and the boundary values.

There appear to be many further applications of this inequality.

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DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UTAH 84601