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COEFFICIENTS FOR THE AREA THEOREM

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ABSTRACT. Let $f(z)=\sum_{n=1}^{\infty} a_n z^n$, and set $G(z)=f(z^{-p})^{-1/p}=\sum_{n=0}^{\infty} g_{np-1} z^{1-np}$. This paper finds an explicit formula for g_{np-1} in terms of the a_n. Such a formula (apparently previously unknown) may be very useful in the theory of univalent functions.

1. **Introduction.** The importance of the area theorem in the theory of univalent functions is well known [1]. One form of this theorem [8, p. 209] runs as follows.

THEOREM A. *Suppose that $f(z)$ is regular and univalent in $|z|<1$, and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Define $G(z)$ by

$$G(z) = \frac{1}{f(1/z)} = z + \sum_{n=0}^{\infty} \frac{g_n}{z^n}. \tag{2}$$

Then

$$\sum_{n=1}^{\infty} n\,|g_n|^2 \leqq 1. \tag{3}$$

Although the inequality (3) is useful in obtaining properties of the function $f(z)$ and domains for the coefficients a_n, this usefulness is somewhat limited because the g_n are complicated functions of the a_n, and the computation of these functions is time consuming for large n (here large means $n>4$). As far as the author is aware no general formula for g_n has been given up to this time. The purpose of this paper is to present such a formula.

Although our methods are different from those used by Hummel [4], it was his excellent treatment of an analogous problem in determining the Grunsky coefficients that encouraged the author to search for the formula given below.

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It should be mentioned that the inequality (3) is quite often misnamed. A study of the papers involved [2], [3] clearly shows that (3) was first proved by Gronwall, and consequently we refer to it as Gronwall's inequality. This priority of Gronwall has also been noticed by Jenkins [5, p. 2].

2. **The formula.** It is often useful to consider $(f(z^2))^{1/2}$ rather than $f(z)$. More generally we let p be an arbitrary positive integer, and we replace (2) by

$$G(z) = (f(1/z^p))^{-1/p} = z + \sum_{n=1}^{\infty} \frac{g_{np-1}^{(p)}}{z^{np-1}}. \tag{4}$$

Direct and laborious computations give the following formulas

$$g_{p-1}^{(p)} = -\frac{1}{p} a_2, \tag{5}$$

$$g_{2p-1}^{(p)} = -\frac{1}{p} a_3 + \frac{p+1}{2p^2} a_2^2, \tag{6}$$

$$g_{3p-1}^{(p)} = -\frac{1}{p} a_4 + \frac{p+1}{p^2} a_2 a_3 - \frac{(2p+1)(p+1)}{6p^3} a_2^3, \tag{7}$$

$$\begin{aligned} g_{4p-1}^{(p)} = &-\frac{1}{p} a_5 + \frac{p+1}{p^2} a_2 a_4 + \frac{p+1}{2p^2} a_3^2 \\ &- \frac{(2p+1)(p+1)}{2p^3} a_2^2 a_3 + \frac{(3p+1)(2p+1)(p+1)}{24p^4} a_2^4. \end{aligned} \tag{8}$$

Before proceeding, it is convenient to introduce a compact notation for the products that are beginning to appear in (7) and (8). Indeed we set

$$\gamma(p, m) = (p+1)(2p+1) \cdots (mp+1) = \prod_{j=1}^{m} (jp+1), \tag{9}$$

with the usual convention that $\gamma(p, 0) = 1$. Then

$$\begin{aligned} g_{5p-1}^{(p)} = &-\frac{1}{p} a_6 + \frac{p+1}{p^2} a_2 a_5 + \frac{p+1}{p^2} a_3 a_4 \\ &- \frac{\gamma(p,2)}{2p^3} (a_2^2 a_4 + a_2 a_3^2) + \frac{\gamma(p,3)}{6p^4} a_2^3 a_4 - \frac{\gamma(p,4)}{120p^5} a_2^5, \end{aligned} \tag{10}$$

$$
\begin{aligned}
g_{6p-1}^{(p)} = & -\frac{1}{p}a_7 + \frac{p+1}{p^2}\left(a_2a_6 + a_3a_5 + \frac{a_4^2}{2}\right) \\
& - \frac{\gamma(p,2)}{p^3}\left(\frac{a_2^2a_5}{2} + a_2a_3a_4 + \frac{a_3^3}{6}\right) \\
& + \frac{\gamma(p,3)}{p^4}\left(\frac{a_2^3a_4}{6} + \frac{a_2^2a_3^2}{4}\right) \\
& - \frac{\gamma(p,4)}{24p^5}a_2^4a_3 + \frac{\gamma(p,5)}{720p^6}a_2^6.
\end{aligned} \tag{11}
$$

We have listed the first six cases of our general formula explicitly because the sixth case, equation (11), is the first one that involves a_7, and 7 is the smallest integer for which the general conjecture, $|a_n| \leqq n$, is currently open.

The general formula is almost obvious from the six cases already cited. However to express it in a simple way it is convenient to set $b_n = a_{n+1}$, for $n = 1, 2, \cdots$, in order to bring to the surface a certain order in the formulas that might otherwise go unnoticed (see Hummel [4]).

THEOREM 1. *Let $S(n)$ be the set of all n-tuples $(r_1, r_2, \cdots, r_n)$ of nonnegative integers for which*

$$r_1 + 2r_2 + 3r_3 + \cdots + nr_n = n, \tag{12}$$

and for each such n-tuple define m by

$$r_1 + r_2 + \cdots + r_n = m. \tag{13}$$

If $b_n = a_{n+1}$, $n = 1, 2, \cdots$, and $g_{np-1}^{(p)}$ is defined by equation (4) *where $f(z)$ is given by* (1), *then*

$$g_{np-1}^{(p)} = \sum \frac{(-1)^m \gamma(p, m-1) b_1^{r_1} b_2^{r_2} \cdots b_n^{r_n}}{p^m r_1! r_2! \cdots r_n!}, \tag{14}$$

where the sum is over all n-tuples in $S(n)$.

It is a simple matter to check that the formula (14) gives (5), (6), (7), (8), (10) and (11) when $n = 1, 2, 3, 4, 5$, and 6 respectively.

PROOF. For simplicity we drop the superscript on g. We differentiate the identity (4) and then set $1/z^p = \zeta$. After a few minor steps we find that

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \frac{G(z)}{z} = \sum_{n=0}^{\infty} (1 - np) g_{np-1} \zeta^n, \tag{15}$$

with $g_{-1}\equiv 1$. The power series for f, f', and G transform (15) into

$$(16)\qquad \left(\sum_{n=1}^{\infty} na_n\zeta^n\right)\left(\sum_{n=0}^{\infty} g_{np-1}\zeta^n\right) = \left(\sum_{n=1}^{\infty} a_n\zeta^n\right)\left(\sum_{n=0}^{\infty}(1-np)g_{np-1}\zeta^n\right).$$

For fixed integer $n\geqq 1$ we equate coefficients of ζ^{n+1} in (16) and find

$$(17)\qquad \sum_{k=0}^{n}(n-k+kp)a_{n+1-k}g_{kp-1} = 0.$$

Since $a_1=1$, we can solve (17) for g_{np-1}. This gives

$$(18)\qquad g_{np-1} = -\frac{1}{np}\sum_{k=0}^{n-1}(n-k+kp)a_{n+1-k}g_{kp-1}.$$

Now, equation (18) is a recursion formula that allows us to compute g_{np-1} from the ones with smaller index, and as such determines the sequence of coefficients g_{np-1} in a unique manner. Consequently, in order to prove Theorem 1, it is sufficient to show that for each index n the coefficients g_{np-1} defined by equation (14) do indeed satisfy (18). Again it is an easy matter to check that if we set $n=1, 2, 3, 4, 5$, and 6 in (18) this formula does give the first six coefficients already listed.

As already indicated we replace a_{n+1-k} by b_{n-k}. Then (18) becomes

$$(19)\qquad g_{np-1} = -\frac{1}{np}\sum_{k=0}^{n-1}(n-k+kp)b_{n-k}g_{kp-1}.$$

We proceed by induction. That is, we assume that for each $k=1, 2, \cdots, n-1$,

$$(20)\qquad g_{kp-1} = \sum \frac{(-1)^j\gamma(p, j-1)b_1^{r_1}b_2^{r_2}\cdots b_k^{r_k}}{p^j r_1! r_2! \cdots r_k!},$$

where $j\equiv r_1+r_2+\cdots+r_k$ and the sum is over $S(k)$ the set of all nonnegative k-tuples $(r_1, r_2, \cdots, r_k)$ for which $r_1+2r_2+\cdots+kr_k=k$. Now if $k<n$, it does no harm to enlarge the k-tuple to an n-tuple by adjoining a suitable number of zeros. On the one hand any solution of

$$(21)\qquad r_1+2r_2+\cdots+nr_n=k, \qquad k<n,$$

in nonnegative integers must give $r_i=0$, if $i=k+1, k+2, \cdots, n$. On the other hand the inclusion of the factors $b_i^{r_i}/r_i!$ in (20) causes no harm because for $i=k+1, k+2, \cdots, n$ these factors are 1. Consequently (20) can be replaced by

$$(22)\qquad g_{kp-1} = \sum \frac{(-1)^j\gamma(p, j-1)b_1^{r_1}b_2^{r_2}\cdots b_n^{r_n}}{p^j r_1! r_2! \cdots r_n!}$$

where $k \leqq n$, $j \equiv r_1+r_2+\cdots+r_n$, and the sum is over the set $S(k)$ of all nonnegative integer solutions of (21). We use (22) in the right side of (19). Then for the right side R, we have

$$R \equiv -\frac{1}{np}\sum_{k=0}^{n-1}\sum_{S(k)}\frac{(-1)^j(n-k+kp)\gamma(p,j-1)b_{n-k}b_1^{r_1}b_2^{r_2}\cdots b_n^{r_n}}{p^j r_1!r_2!\cdots r_n!}. \tag{23}$$

Now let $(s_1, s_2, \cdots, s_n)$ be any fixed n-tuple in $S(n)$, so that

$$\sum_{i=1}^{n} is_i = n, \qquad \sum_{i=1}^{n} s_i = m. \tag{24}$$

We are to determine the coefficient C of $b_1^{s_1}b_2^{s_2}\cdots b_n^{s_n}$ in (23). This coefficient may arise from combining several terms from the sum and in fact such terms arise if and only if $b_{n-k}b_1^{r_1}b_2^{r_2}\cdots b_n^{r_n}=b_1^{s_1}b_2^{s_2}\cdots b_n^{s_n}$. To be specific let α be an index for which $s_\alpha \geqq 1$, and let $r_i=s_i$ if $i\neq\alpha$, and let $r_\alpha=s_\alpha-1$. For this fixed α, we have $j=\sum_{i=1}^n r_i=m-1$. In (23) we set $n-k=\alpha$. If A is the set of α for which $s_\alpha\neq 0$, then

$$C = -\frac{1}{np}\sum_{\alpha\in A}\frac{(-1)^{m-1}(\alpha+(n-\alpha)p)\gamma(p,m-2)}{p^{m-1}r_1!r_2!\cdots r_n!}. \tag{25}$$

Inserting the factor s_α in the numerator and denominator of (25) we have

$$\begin{aligned} C &= \sum_{\alpha\in A}\frac{(-1)^m s_\alpha(\alpha+(n-\alpha)p)\gamma(p,m-2)}{np^m s_1!s_2!\cdots s_n!} \\ &= \frac{(-1)^m\gamma(p,m-2)}{np^m s_1!s_2!\cdots s_n!}\sum_{\alpha\in A} s_\alpha(\alpha+(n-\alpha)p). \end{aligned} \tag{26}$$

But if $s_\alpha=0$, the corresponding term in the sum is zero, hence (using (24))

$$\begin{aligned} C &= \frac{(-1)^m\gamma(p,m-2)}{np^m s_1!s_2!\cdots s_n!}\sum_{\alpha=1}^{n}(\alpha s_\alpha + nps_\alpha - p\alpha s_\alpha) \\ &= \frac{(-1)^m\gamma(p,m-2)}{np^m s_1!s_2!\cdots s_n!}(n+npm-pn) \\ &= \frac{(-1)^m\gamma(p,m-2)}{p^m s_1!s_2!\cdots s_n!}((m-1)p+1) = \frac{(-1)^m\gamma(p,m-1)}{p^m s_1!s_2!\cdots s_n!}. \end{aligned} \tag{27}$$

But this is precisely the coefficient of $b_1^{s_1}b_2^{s_2}\cdots b_n^{s_n}$ required in formula (14). Since the argument holds for each fixed $(s_1, s_2, \cdots, s_n)$ this completes the proof of Theorem 1.

3. **Remarks.** Although the formula for $g_{n,p-1}$ was derived for p a positive integer, and it was assumed that $f(z)$ is univalent in $|z|<1$, the formula is independent of both of these assumptions. The only requirement on $f(z)$ is that $f(z)$ have a simple zero at $z=0$ and $f'(0)=1$. The power series (4) for $G(z)$ will converge for $|z|>R_0$ where $R_0=1/r_0$ and r_0 is the modulus of the smallest zero of $f(z)/z$. Further p may be any integer positive or negative, as long as $p\neq 0$. We can obtain numerical checks on the formula, and deduce various identities by selecting special values for p and special functions. For example if $p=-1$, then $G(z)\equiv f(z)$. This explains the presence of the factor $p+1$ in every term except the term $-a_n/p$ in (14).

Suppose that $p=-2$. Then

$$G(z) = (f(z^2))^{1/2} \equiv z + \sum_{n=1}^{\infty} c_{2n+1}z^{2n+1}, \tag{28}$$

and (14) gives

$$c_{2n+1}=\frac{1}{2}a_{n+1} + \sum_{S(n);\,m>1} \frac{(-1)^{m+1}(2m-3)!a_2^{r_1}a_3^{r_2}\cdots a_{n+1}^{r_n}}{2^{2m-2}(m-2)!r_1!r_2!\cdots r_n!}, \tag{29}$$

a formula for the coefficients in the square root of a power series.

The case $p=1$ gives a formula for the reciprocal of a power series. Thus if

$$g(z) \doteq \frac{1}{f(z)} = \frac{1}{z} + \sum_{n=0}^{\infty} d_n z^n, \tag{30}$$

then for $n\geqq 1$,

$$d_{n-1} = \sum_{S(n)} \frac{(-1)^m m!a_2^{r_1}a_3^{r_2}\cdots a_{n+1}^{r_n}}{r_1!r_2!\cdots r_n!}. \tag{31}$$

It would be convenient to have a formula that would give the number of elements in the set $S(n)$. However, this is too much to expect because this number is $p(n)$, the number of unrestricted partitions of n. The function $p(n)$ has been the subject of intensive research since the days of Euler, and although much is known [6], [7], a simple formula for $p(n)$ has not been found and it is doubtful if such a formula exists. Many of the properties of $p(n)$ follow from the relation

$$\frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)\cdots} = 1 + \sum_{n=1}^{\infty} p(n)z^n. \tag{32}$$

Values of $p(n)$ have been computed for $n\leqq 1000$, and may be found in the Royal Society Mathematics Tables, vol. 4, Cambridge Univ. Press, London, 1958, pp. 118–121.

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