SHORTER NOTES

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ON LIAPUNOV'S INEQUALITY

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ABSTRACT. An elementary proof is given of the Liapunov inequality $\int_a^b p_+(x) dx > 4/(b-a)$.

The purpose of this note is to give an elementary proof of a well-known inequality due to Liapunov [3] which states that if $p(x)$ is continuous on the interval $I: [a, b]$ and if there exists a solution $y(x) \not\equiv 0$ of the differential equation²

$$(1) \quad y'' + p(x)y = 0$$

that has two zeros on I , then

$$(2) \quad \int_a^b p_+(x) dx > \frac{4}{b-a}.$$

Here, $p_+(x) = \frac{1}{2}[p(x) + |p(x)|]$.

Consider the differential equation

$$(1)' \quad y'' + p_+(x)y = 0.$$

Inasmuch as $p_+(x) \geq p(x)$ on $[a, b]$, there exists a solution $y(x)$ of (1)' such that $y(a) = y(c) = 0$, with $y(x) > 0$ on $(a, c) \in I$. Let $x = f$ be a point on (a, c) at which $y(x)$ attains its absolute maximum. We note that since $y''(x) \leq 0$ on (a, c) , $y(x)$ possesses no relative minima on that (open) interval (it may possess a line of relative maxima, however). Draw the two chords that connect the point $[f, y(f)]$ with the points $(a, 0)$ and $(c, 0)$.

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² Nehari [5] has extended Liapunov's inequality to a linear differential equation of order n . See also [1], [4, p. 271], [2, p. 74] and the generalization of Reid [6].

Then, a reference to a figure makes it plain that

$$y'(a) > \frac{y(f)}{f-a} \quad \text{and} \quad -y'(c) > \frac{y(f)}{c-f} \geq \frac{y(f)}{b-f}.$$

It follows that

$$(3) \quad -\frac{y'(c) - y'(a)}{y(f)} > \frac{b-a}{(f-a)(b-f)};$$

that is,

$$-\frac{1}{y(f)} \int_a^c y''(x) dx = \frac{1}{y(f)} \int_a^c p_+(x)y(x) dx > \frac{b-a}{(f-a)(b-f)}.$$

Inasmuch as

$$\frac{1}{y(f)} \int_a^c p_+(x)y(x) dx < \int_a^c p_+(x) dx,$$

we have

$$(4) \quad \int_a^b p_+(x) dx \geq \int_a^c p_+(x) dx > \frac{b-a}{(f-a)(b-f)}.$$

But the last member of (4) is readily seen to be greater than or equal to $4/(b-a)$, with equality only if f is the midpoint of (a, b) . The proof is complete.

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