AN EIGENVALUE PROBLEM FOR THE STIELTJES MEAN SIGMA-INTEGRAL RELATED TO GRONWALL'S INEQUALITY

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ABSTRACT. This paper is concerned with the solution of the eigenvalue problem $\lambda f(t) = \int_0^t f(s) dg(s)$, $0 \leq t \leq T$, where g is non-decreasing and right continuous on $[0, T]$ and the integral is the Stieltjes mean sigma-integral. In addition to the solution of the general problem, the solution of the problem with positive eigenfunctions is given. The relationship of the eigenvalue problem to the failure of the Gronwall inequality for g is also established. Furthermore, the techniques used to solve this eigenvalue problem are readily adapted to solve the extended problem in which the function g is merely of bounded variation and right continuous.

1. Introduction. A Gronwall inequality for the Stieltjes mean sigma-integral was established by W. W. Schmaedeke and G. R. Sell in [2]. In the last paragraph of that paper, the authors indicated that the failure of the inequality could be related to an eigenvalue problem and then indicated a characterization of the eigenvalues of the problem. This characterization is our Theorem 2 below for which we present a first proof. We indicate what the relationship is between the failure of the Gronwall inequality and the eigenvalue problem in our corollary to Theorem 2. In proceeding to establish Theorem 2, we first consider the eigenvalue problem for non-negative eigenfunctions, Theorem 1, and then suitably modify our constructive solution of that problem.

2. Terminology and notation. Throughout this paper suppose that g is a nondecreasing right continuous function on the interval $[0, T]$. We let $\Delta g(t) = g(t) - g(t^-)$. Let F be the set of all quasi-continuous functions on $[0, T]$ which are right continuous, and let F^+ consist of those members of F which are nonnegative valued. For reference we list a slightly extended

Received by the editors November 30, 1970 and, in revised form, September 14, 1971.

AMS 1970 subject classifications. Primary 45C05, 26A39; Secondary 26A84, 47A10.

Key words and phrases. Stieltjes mean sigma-integral, Gronwall inequality, eigenvalue problem.

¹ This paper is based in part on the recent M.A. thesis of the first author, supervised by the second author.

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version of the Gronwall inequality for the mean sigma-integral, which is also established by the proof in [2].

THEOREM. *If there exists a t in $(0, T]$ such that $\Delta g(t) \geq 2$, let T' be the least such t ; otherwise let $T' = T$. Suppose $\varepsilon \geq 0$. Then there exists a K , $K \geq 0$, such that if f is in F^+ and*

$$f(t) \leq \varepsilon + \int_0^t f(s) dg(s), \quad 0 \leq t \leq T,$$

then

$$f(t) \leq K, \quad 0 \leq t < T'.$$

Also for reference we list the equality

$$\lim_{a \rightarrow b^-} \int_a^b f(s) dg(s) = \frac{1}{2}[f(b) + f(b^-)] \Delta g(b).$$

We define the operator U corresponding to the function g by the equation

$$Uf(t) = \int_0^t f(s) dg(s), \quad 0 \leq t \leq T.$$

3. Statements of theorems. We can now list the three results we shall establish in this paper and a lemma which we shall use in proving the first theorem.

THEOREM 1. $\lambda > 0$ is an eigenvalue of U with a nontrivial eigenfunction in F^+ if and only if there exists a T^* in $(0, T]$ such that $\Delta g(T^*) = 2\lambda$ and $\Delta g(t) < 2\lambda$ if $T^* < t \leq T$.

THEOREM 2. For $\lambda > 0$, λ is an eigenvalue of U with a nontrivial eigenfunction in F if and only if there exists T^* in $(0, T]$ such that $\Delta g(T^*) = 2\lambda$.

COROLLARY 1. U has an eigenvalue λ ($\lambda \geq 1$) if and only if the Gronwall inequality for g fails at T' .

LEMMA 1. Suppose that there exists a sequence of functions g_n of bounded variation converging uniformly to g on $[0, T]$, with each $V_0^T g_n \leq V_0^T g$. Let $\lambda = 1$ be an eigenvalue for each operator U_n where

$$U_n f(t) = \int_0^t f(s) dg_n(s), \quad 0 \leq t \leq T.$$

If a sequence of corresponding eigenfunctions f_n converges uniformly to a quasi-continuous function f on $[0, T]$, then f is an eigenfunction for the operator U corresponding to $\lambda = 1$.

4. Proof of theorems. We do not give the proof of Lemma 1 here, but comment that it is easily proved using Lemma 4.2a of [1]. Without loss of generality we give the proofs of Theorems 1 and 2 for $\lambda=1$.

PROOF OF THEOREM 1. Suppose that f in F^+ is a nontrivial eigenfunction for U . Using Lemma 1 of [2] we see that since $f(0)=0$, there is a $t^*>0$ such that $f(t)=0$ for $0\leq t<t^*$ and that if T^* is the supremum of all such t^* 's then $f(T^*)>0$. From the equality $f(T^*)=\int_0^{T^*} f(s) dg(s)$, we conclude that $\Delta g(T^*)=2$. Since g is nondecreasing, $f(t)\geq f(T^*)$ if $T^*\leq t\leq T$. There cannot exist a t' in $(T^*, T]$ such that $\Delta g(t')>2$, for then we would have

$$f(t') = f(t'^-) \cdot (1 + \frac{1}{2} \Delta g(t')) / (1 - \frac{1}{2} \Delta g(t'))$$

which would imply that $f(t')<0$. This ends the necessity part of the proof.

For the sufficiency part, let t_0 be the largest t in $[0, T]$ such that $\Delta g(t)=2$ and let $T''=\inf\{t \text{ in } (t_0, T]: g \text{ is constant on } (t, T]\}$. Now, there exists a sequence of functions g_n on $[0, T]$ converging uniformly to g such that for each positive integer n , g_n agrees with g off of $[t_0, T'']$, and on $[t_0, T'']$ g_n is a step function such that

- (i) $|g_n(t)-g(t)|<1/n$ if $t_0\leq t<T''$;
- (ii) $\Delta g_n(t)<2$ for $t\in(t_0, T'']$;
- (iii) at each point t' of discontinuity of g_n , $g_n(t')=g(t')$;
- (iv) if g_n is discontinuous at t' and $l>n$ then g_l is discontinuous at t' and $g_l(t')=g_n(t')$.

We let $h_n(t)=(1+\frac{1}{2}\Delta g_n(t))/(1-\frac{1}{2}\Delta g_n(t))$, and define

$$\begin{aligned} f_n(t) &= 0, & 0 \leq t < t_0 \\ &= \prod_{s \in [t_0, t]} h_n(s), & t_0 \leq t \leq T. \end{aligned}$$

f_n is a nonnegative valued eigenfunction for U_n determined by g_n . Note that $\{f_n(T'')\}_{n=1}^\infty$ is a decreasing sequence bounded below by 1. Finally, we shall show that the sequence of eigenfunctions f_n converges uniformly on $[0, T]$. (The limit function f will then be in F^+ .) Let

$$b_n = \min\{1 - \frac{1}{2}\Delta g_n(t): t_0 < t \leq T''\},$$

then $b \equiv b_1$ is a lower bound for the set of all b_n 's. Also let $B=f_1(T'')$.

Suppose $\varepsilon>0$. Let $\varepsilon^*=b \ln(1+\varepsilon/B)$. Choose N such that $|g_n-g|<\varepsilon^*$ and $|f_n(T'')-f_N(T'')|<\varepsilon$ if $n>N$. Let $t_0<t_1<\cdots<t_p=T''\leq T$ be the end points of the intervals of constancy of g_n on $[t_0, T]$. Suppose $t\in[t_i, t_{i+1})$. We consider two cases.

Case (i). $f_n(t)>f_N(t)$. Then

$$f_n(t) = f_n(t_i) \cdot \prod_{s \in [t_i, t]} h_n(s) < f_N(t_i) \cdot \exp(\varepsilon^*/b),$$

so that

$$\begin{aligned} f_n(t) - f_N(t) &< (\exp(\varepsilon^*/b) - 1) \cdot f_N(t) \\ &= \varepsilon \cdot f_N(t) \cdot B^{-1} < \varepsilon. \end{aligned}$$

Case (ii). $f_N(t) > f_n(t)$. Here, if $[t_j, t_{j+1})$ is an interval of constancy of g_N and the end points of the intervals of constancy of g_n on $[t_j, t_{j+1})$ are $t_j = t_{1j} < t_{2j} < \cdots < t_{lj} = t_{j+1}$, then

$$\Delta g_N(t_{j+1}) = \Delta g_n(t_{2j}) + \cdots + \Delta g_n(t_{lj})$$

and

$$h_N(t_{j+1}) > h_n(t_{2j}) \cdots h_n(t_{lj}).$$

Thus

$$\begin{aligned} f_N(t) - f_n(t) &\leq f_N(t_i) - f_n(t_i) \\ &< (f_N(t_i) - f_n(t_i)) \cdot h_N(t_{i+1}) \cdots h_N(T^*) \\ &\leq f_N(T'') - f_n(T'') < \varepsilon. \end{aligned}$$

PROOF OF THEOREM 2. With slight natural modifications, our proof of Theorem 1 yields the proof of Theorem 2. In the sufficiency part for instance, the g_n 's can be constructed so that additionally each one agrees with g at each t in $(t_0, T]$ where $\Delta g(t) > 2$; then on the successive sub-intervals of $[0, T]$ with such t 's for end points, each f_n will take on alternately negative and positive values with the $|f_n|$'s behaving as the f_n 's did in the proof of Theorem 1. In the necessity part, if f is locally nonpositive valued at 0, apply the argument of the necessity proof of Theorem 1 to $-f$.

PROOF OF COROLLARY 1. If $\lambda \geq 1$ is an eigenvalue of U , then by Theorem 2, there is a T^* in $(0, T]$ such that $\Delta g(T^*) = 2\lambda$. Consequently $\Delta g(T') \geq 2$, so that the Gronwall inequality for g fails at T' as is shown by the example in [2]. On the other hand, if the Gronwall inequality for g fails at T' , Lemma 3 of [2] can be used to show that $\Delta g(T') \geq 2$ which then implies that $\lambda = \frac{1}{2}\Delta g(T')$ is an eigenvalue of U .

5. Extension of the eigenvalue problem. Now let g' be a function which is merely of bounded variation and right continuous on $[0, T]$ and let

$$U'f(t) = \int_0^t f(s) dg'(s), \quad 0 \leq t \leq T.$$

The referee has pointed out that the eigenvalue problem for U' can be handled with little additional effort. Of course, $\lambda = 0$ is always an eigenvalue of U' having corresponding eigenfunctions in F^+ . Using the technique of proof of Theorem 1 again, we get the result that $\lambda < 0$ is an eigenvalue of U' with a nontrivial eigenfunction in F^+ if and only if there exists a T^* in $(0, T]$ such that $\Delta g'(T^*) = 2\lambda$ and $2\lambda < g'(t) \leq -2\lambda$ if $T^* < t \leq T$, and we

get a dual result for $\lambda > 0$. In general, we have that for λ a real number, λ is an eigenvalue of U' with a nontrivial eigenfunction in F if and only if there is a T^* in $(0, T]$ such that $\Delta g'(T^*) = 2\lambda$.

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