A RANDOM FREDHOLM INTEGRAL EQUATION

W. J. PADGETT AND CHRIS P. TSOKOS

ABSTRACT. The aim of this paper is the study of a random or stochastic integral equation of the Fredholm type given by $x(t; \omega) = h(t; \omega) + \int_0^\infty k_0(t, \tau; \omega)e(\tau, x(\tau; \omega)) d\tau$, $t \geq 0$, where $\omega \in \Omega$, the supporting set of the probability measure space $(\Omega, \mathcal{A}, P)$. The existence and uniqueness of a random solution to the above stochastic integral equation is considered. A random solution, $x(t; \omega)$, of such a random equation is defined to be a random function which satisfies the equation almost surely. Several theorems and useful special cases are presented which give conditions such that a random solution exists.

1. Introduction. The object of this note is to state some results concerning the existence and uniqueness of a *random solution* of a stochastic integral equation of the Fredholm type of the form

$$(1.1) \quad x(t; \omega) = h(t; \omega) + \int_0^\infty k_0(t, \tau; \omega)e(\tau, x(\tau; \omega)) d\tau,$$

where $t \in R_+$ and

(i) $\omega \in \Omega$, where Ω is the supporting set of the complete probability measure space $(\Omega, \mathcal{A}, P)$;

(ii) $x(t; \omega)$ is the unknown random function for $t \in R_+$, the nonnegative real numbers;

(iii) $h(t; \omega)$ is the stochastic free term defined for $t \in R_+$;

(iv) $k_0(t, \tau; \omega)$ is the stochastic kernel defined for t and τ in R_+ ; and

(v) $e(t, x)$ is a scalar function defined for $t \in R_+$ and $x \in R$, the real line.

The above equation is a generalization of the equation of Anderson [1] in that the stochastic kernel is a general random function, the equation is nonlinear, and the interval of integration is R_+ . To study existence of a unique random solution of equation (1.1), we first present theorems concerning the existence of a unique random solution of the mixed

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random Volterra-Fredholm equation of the form

$$(1.2) \quad \begin{aligned} x(t; \omega) = & h(t; \omega) + \int_0^t k(t, \tau; \omega) f(\tau, x(\tau; \omega)) d\tau \\ & + \int_0^\infty k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau, \quad t \geq 0, \end{aligned}$$

where

(vi) $k(t, \tau; \omega)$ is a stochastic kernel defined for t and τ satisfying $0 \leq \tau \leq t < \infty$; and

(vii) $f(t, x)$ is a scalar function of $t \in R_+$ and $x \in R$.

Then equation (1.1) is obtained from (1.2) when $k(t, \tau; \omega)$ is equal to zero for all t and τ such that $0 \leq \tau \leq t < \infty$ and almost all $\omega \in \Omega$. Also, the stochastic integral equation of the Volterra type of Tsokos [9] may be obtained from (1.2).

Certain aspects of the "theory of admissibility" as introduced into the theory of deterministic integral equations by Corduneanu [2] and methods of probabilistic functional analysis were used in obtaining the results announced here. Nonstochastic integral equations of the form (1.2) have been studied by Miller, Nohel, and Wong [5], Petrovanu [7], and Corduneanu [3], among others.

2. Definitions and assumptions. We make the following assumptions concerning the equations (1.1) and (1.2). The random functions $x(t; \omega)$ and $h(t; \omega)$ are second order stochastic processes, continuous (in mean square) in $t \geq 0$; that is, for each $t \in R_+$, $x(t; \omega)$ and $h(t; \omega)$ are in the space $L_2(\Omega, A, P)$. The functions $f(t, x(t; \omega))$ and $e(t, x(t; \omega))$ are functions of $t \in R_+$ with values in $L_2(\Omega, A, P)$. For each t and τ in R_+ , $k_0(t, \tau; \omega)$ has values in the space $L_\infty(\Omega, A, P)$, and the function $k(t, \tau; \omega)$ for each t and τ such that $0 \leq \tau \leq t < \infty$ has values in $L_\infty(\Omega, A, P)$. Thus, $k(t, \tau; \omega)f(\tau, x(\tau; \omega))$, $0 \leq \tau \leq t < \infty$, and $k_0(t, \tau; \omega)e(\tau, x(\tau; \omega))$, $t, \tau \in R_+$, are in $L_2(\Omega, A, P)$.

The mappings

$$(t, \tau) \rightarrow k(t, \tau; \omega) \quad \text{and} \quad (t, \tau) \rightarrow k_0(t, \tau; \omega)$$

from the sets

$$\Delta = \{(t, \tau): 0 \leq \tau \leq t < \infty\}$$

and

$$\Delta_0 = \{(t, \tau): 0 \leq \tau < \infty, 0 \leq t < \infty\},$$

respectively, into $L_\infty(\Omega, A, P)$ are continuous. That is,

$$\text{P-ess sup}_{\omega \in \Omega} |k(t_n, \tau_n; \omega) - k(t, \tau; \omega)| \rightarrow 0$$

and

$$\text{P-ess sup}_{\omega \in \Omega} |k_0(t_n, \tau_n; \omega) - k_0(t, \tau; \omega)| \rightarrow 0$$

as $n \rightarrow \infty$ whenever $(t_n, \tau_n) \rightarrow (t, \tau)$ as $n \rightarrow \infty$. Denote the norm of $k_0(t, \tau; \omega)$ in $L_\infty(\Omega, A, P)$ for $t, \tau \in R_+$ by

$$\| \| k_0(t, \tau; \omega) \| \| = \text{P-ess sup}_\omega |k_0(t, \tau; \omega)|,$$

and the norm of $k(t, \tau; \omega)$ in $L_\infty(\Omega, A, P)$ is denoted similarly. Assume that for each $t \geq 0$, $\| \| k_0(t, \tau; \omega) \| \|$ and $\| \| k_0(t, \tau; \omega) \| \| \cdot \| x(\tau; \omega) \|_{L_2(\Omega, A, P)}$ are integrable with respect to $\tau \in R_+$ for every $x(\tau; \omega) \in C_c$.

DEFINITION 2.1. We shall denote by C_g the space of all continuous functions from R_+ into $L_2(\Omega, A, P)$ such that, for $g(t)$ a positive, continuous function on R_+ and some constant $Z > 0$, we have

$$\left\{ \int_\Omega |x(t; \omega)|^2 dP(\omega) \right\}^{1/2} \leq Zg(t), \quad t \geq 0.$$

The norm in C_g is defined by

$$\| x(t; \omega) \|_{C_g} = \sup_{t \in R_+} \left\{ \frac{\| x(t; \omega) \|_{L_2(\Omega, A, P)}}{g(t)} \right\}.$$

DEFINITION 2.2. Let C denote the space of all continuous and bounded functions from R_+ into $L_2(\Omega, A, P)$.

DEFINITION 2.3. By $C_c = C_c(R_+, L_2(\Omega, A, P))$ we mean the space of all continuous functions from R_+ into $L_2(\Omega, A, P)$ with the topology of uniform convergence on every interval $[0, Q]$, $Q > 0$.

Let B and D be a pair of Banach spaces such that $B, D \subset C_c$, and let T be a linear operator from C_c into itself.

DEFINITION 2.4. The pair of Banach spaces (B, D) is said to be *admissible* with respect to $T: C_c \rightarrow C_c$ if and only if $T(B) \subset D$.

DEFINITION 2.5. The Banach space B is said to be *stronger* than C_c if every convergent sequence in B with respect to its norm also converges in C_c , but the converse is not true in general.

Let $H \subset C_c$ be the space of all functions such that the inner product in $L_2(\Omega, A, P)$ of an element $x(t; \omega)$ with itself is integrable on R_+ , i.e.

$$(2.1) \quad \| x(t; \omega) \|_{L_2(\Omega, A, P)}^2 = \int_\Omega |x(t; \omega)|^2 dP(\omega), \quad t \in R_+,$$

and $y(t; \omega) \in H$ if its inner product with itself and with any $x(t; \omega)$ for which (2.1) is integrable on R_+ is integrable with respect to t , and let H_1 and H_2 be Hilbert spaces contained in H with norms defined by

$$\| x(t; \omega) \|_{H_i} = \left\{ \int_0^\infty \| x(t; \omega) \|_{L_2(\Omega, A, P)}^2 dt \right\}^{1/2}, \quad i = 1, 2.$$

By a *random solution* of a stochastic integral equation such as equation (1.1) we shall mean a random function $x(t; \omega)$ which satisfies the equation *P*-a.e. In order to prove the following results, we employ the well-known fixed point theorems of Krasnosel'skiĭ, Banach, and Schauder [4].

3. Main results. We now present theorems concerning the existence and uniqueness of random solutions of equations (1.1) and (1.2) which are the main results of the paper. We first state two lemmas which will be needed in the proofs.

LEMMA 3.1 (TSOKOS [9]). *Let T be a continuous linear operator from $C_c(R_+, L_2(\Omega, A, P))$ into itself. If B and D are Banach spaces stronger than $C_c(R_+, L_2(\Omega, A, P))$ and if the pair (B, D) is admissible with respect to T , then T is continuous from B into D .*

The following lemma is similar to a theorem appearing in Schmeidler [8] and Yosida [10].

LEMMA 3.2. *The integral operator W_M defined from the Hilbert space H_2 into the Hilbert space H_1 by*

$$(3.1) \quad (W_M x)(t; \omega) = \int_0^M k_0(t, \tau; \omega) x(\tau; \omega) d\tau,$$

for $t \in [0, M]$, where M is a positive integer, is a bounded, completely continuous operator if the stochastic kernel $k_0(t, \tau; \omega)$ is such that $\int_0^\infty \int_0^M \|k_0(t, \tau; \omega)\|^2 d\tau dt$ exists and is finite.

The proof follows the technique of Schmeidler [8, p. 45].

We shall first consider the random integral equation of the Fredholm type of the form

$$(3.2) \quad x(t; \omega) = h(t; \omega) + \int_0^M k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau,$$

for $t \in [0, M]$, where M is a positive integer.

THEOREM 3.1. *Consider the random integral equation (3.2) under the following conditions:*

(i) H_1 and H_2 are Hilbert spaces stronger than C_c such that the pair (H_2, H_1) is admissible with respect to the linear operator W_M given by equation (3.1) where $t \in [0, M]$, M is a positive integer, $k_0(t, \tau; \omega)$ behaves as above, and $\int_0^\infty \int_0^\infty \|k_0(t, \tau; \omega)\|^2 d\tau dt$ exists and is finite;

(ii) $x(t; \omega) \rightarrow e(t, x(t; \omega))$ is a continuous operator on the set

$$S = \{x(t; \omega) : x(t; \omega) \in H_1, \|x(t; \omega)\|_{H_1} \leq \rho\},$$

with values in H_2 such that $\|e(t, x(t; \omega))\|_{H_2} \leq \gamma$, where $\gamma > 0$ and $\rho > 0$ are constants;

(iii) $h(t; \omega) \in H_1$.

Then there exists at least one random solution of equation (3.2), provided that $\|h(t; \omega)\|_{H_1} + K_M \gamma \leq \rho$, where K_M is the norm of the operator W_M .

PROOF. Clearly, S is a closed, bounded, convex subset of H_1 , and H_1 and H_2 are Banach spaces, since they are Hilbert spaces.

Define the operator V_M from S into H_1 by

$$(3.3) \quad (V_M x)(t; \omega) = h(t; \omega) + \int_0^M k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau,$$

for $t \in [0, M]$. From condition (i) of the theorem and Lemma 3.2 the operator W_M is a completely continuous operator from H_2 into H_1 . The function $e(t, x(t; \omega))$ is a continuous and bounded mapping from H_1 into H_2 by condition (ii). We may express the operator V_M as the composite $W_M e$ plus h , and therefore, V_M is a completely continuous operator from S into H_1 (Krasnosel'skii [4]).

We must show that $V_M(S) \subset S$. Let $x(t; \omega) \in S$. Taking the norm of equation (3.3), we obtain

$$\begin{aligned} \|(V_M x)(t; \omega)\|_{H_1} &= \left\| h(t; \omega) + \int_0^M k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau \right\|_{H_1} \\ &\leq \|h(t; \omega)\|_{H_1} + K_M \|e(t, x(t; \omega))\|_{H_2} \\ &\leq \|h(t; \omega)\|_{H_1} + K_M \gamma \leq \rho \end{aligned}$$

by condition (ii) and the last hypothesis of the theorem. Hence, $(V_M x)(t; \omega) \in S$.

Therefore, by Schauder's fixed-point theorem, there exists at least one random element $x(t; \omega) \in S$ satisfying

$$\begin{aligned} x(t; \omega) &= (V_M x)(t; \omega) \\ &= h(t; \omega) + \int_0^M k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau \end{aligned}$$

for $t \in [0, M]$, completing the proof.

The sequence of integral operators W_M converges to the operator W from H_2 into H_1 defined by

$$(3.4) \quad (Wx)(t; \omega) = \int_0^\infty k_0(t, \tau; \omega) x(\tau; \omega) d\tau, \quad t \in R_+,$$

as $M \rightarrow \infty$. Then W is completely continuous under the same condition as in (i) of Theorem 3.1. Hence, we have the following theorem.

THEOREM 3.2. Suppose the random integral equation (1.1) satisfies the following conditions:

(i) H_1 and H_2 are Hilbert spaces stronger than C_c such that the pair (H_2, H_1) is admissible with respect to the linear operator W given by

equation (3.4), where $k_0(t, \tau; \omega)$ behaves as described above and $\int_0^\infty \int_0^\infty \|k_0(t, \tau; \omega)\|^2 d\tau dt$ exists and is finite;

(ii) Same as condition (ii) of Theorem 3.1;

(iii) Same as condition (iii) of Theorem 3.1.

Then there exists at least one random solution of equation (1.1) in S , provided $\|h(t; \omega)\|_{H_1} + K\gamma \leq \rho$, where K is the norm of W .

The proof is similar to that of Theorem 3.1.

The following theorem concerns the existence of a *unique* random solution of the random integral equation (1.2).

THEOREM 3.3. Consider the random integral equation (1.2) subject to the following conditions:

(i) B and D are Banach spaces stronger than C_c and (B, D) is admissible with respect to each of the operators

$$(Tx)(t; \omega) = \int_0^t k(t, \tau; \omega)x(\tau; \omega) d\tau$$

and

$$(Wx)(t; \omega) = \int_0^\infty k_0(t, \tau; \omega)x(\tau; \omega) d\tau,$$

for $t \in R_+$, where $k(t, \tau; \omega)$ and $k_0(t, \tau; \omega)$ behave as above;

(ii) $x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator on

$$S = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$$

with values in B satisfying

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \lambda \|x(t; \omega) - y(t; \omega)\|_D$$

for $x(t; \omega), y(t; \omega) \in S$ and λ a constant;

(iii) $x(t; \omega) \rightarrow e(t, x(t; \omega))$ is an operator on S with values in B satisfying

$$\|e(t, x(t; \omega)) - e(t, y(t; \omega))\|_B \leq \xi \|x(t; \omega) - y(t; \omega)\|_D$$

for $x(t; \omega), y(t; \omega) \in S$ and ξ a constant;

(iv) $h(t; \omega) \in D$.

Then there exists a unique random solution of equation (1.2), provided

$$\lambda K_1 + \xi K_2 < 1,$$

$$\|h(t; \omega)\|_D + K_1 \|f(t, 0)\|_B + K_2 \|e(t, 0)\|_B \leq \rho(1 - \lambda K_1 - \xi K_2),$$

where K_1 and K_2 are the norms of T and W , respectively.

PROOF. The operators T and W are continuous operators from B into D by the continuity and integrability assumptions on $k(t, \tau; \omega)$ and

$k_0(t, \tau; \omega)$, condition (i), and Lemma 3.1. Therefore, T and W are bounded operators.

Define the operators U and V from S into D by

$$\begin{aligned}(Ux)(t; \omega) &= h(t; \omega) + \int_0^t k(t, \tau; \omega) f(\tau, x(\tau; \omega)) d\tau, \\ (Vx)(t; \omega) &= \int_0^\infty k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau, \quad t \geq 0.\end{aligned}$$

We show that the conditions of Banach's fixed-point theorem hold. That is, $(Ux)(t; \omega) + (Vx)(t; \omega) \in S$ whenever $x(t; \omega) \in S$, and $U + V$ is a contraction operator on S .

Let $x(t; \omega), y(t; \omega) \in S$. Then

$$\begin{aligned}\|(Ux)(t; \omega) + (Vx)(t; \omega)\|_D &= \left\| h(t; \omega) + \int_0^t k(t, \tau; \omega) f(\tau, x(\tau; \omega)) d\tau \right. \\ &\quad \left. + \int_0^\infty k_0(t, \tau; \omega) e(\tau, x(\tau; \omega)) d\tau \right\|_D \\ &\leq \|h(t; \omega)\|_D + K_1 \|f(t, x(t; \omega))\|_B + K_2 \|e(t, x(t; \omega))\|_B.\end{aligned}$$

But from condition (ii) $\|f(t, x(t; \omega))\|_B \leq \lambda \|x(t; \omega) - 0\|_D + \|f(t, 0)\|_B$, and from condition (iii) a similar inequality holds for $e(t, x(t; \omega))$. Thus,

$$\begin{aligned}\|(Ux)(t; \omega) + (Vx)(t; \omega)\|_D &\leq \|h(t; \omega)\|_D + K_1 \|f(t, 0)\|_B + \lambda K_1 \rho + K_2 \|e(t, 0)\|_B + \xi K_2 \rho \\ &\leq \rho(1 - \lambda K_1 - \xi K_2) + \lambda K_1 \rho + \xi K_2 \rho = \rho\end{aligned}$$

by the last hypothesis of the theorem. So $(Ux)(t; \omega) + (Vx)(t; \omega) \in S$.

Also,

$$\begin{aligned}\|(Ux)(t; \omega) + (Vx)(t; \omega) - (Uy)(t; \omega) - (Vy)(t; \omega)\|_D &= \left\| \int_0^t k(t, \tau; \omega) [f(\tau, x(\tau; \omega)) - f(\tau, y(\tau; \omega))] d\tau \right. \\ &\quad \left. + \int_0^\infty k_0(t, \tau; \omega) [e(\tau, x(\tau; \omega)) - e(\tau, y(\tau; \omega))] d\tau \right\|_D \\ &\leq K_1 \|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \\ &\quad + K_2 \|e(t, x(t; \omega)) - e(t, y(t; \omega))\|_B \\ &\leq (\lambda K_1 + \xi K_2) \|x(t; \omega) - y(t; \omega)\|_D\end{aligned}$$

by condition (ii). Since $\lambda K_1 + \xi K_2 < 1$ by hypothesis, $U + V$ is a contraction mapping on S .

Therefore, by Banach's fixed-point theorem there exists a unique element $x(t; \omega) \in S$ satisfying

$$x(t; \omega) = (Ux)(t; \omega) + (Vx)(t; \omega),$$

that is, a unique random solution of equation (1.2) exists, completing the proof.

For the case that W is the null operator, we obtain the main theorem of Tsokos [9]. For the case that T is the null operator, we obtain equation (1.1), and in this instance the above theorem gives conditions under which a unique random solution of equation (1.1) exists.

Several special cases of the above results may be obtained by taking as the spaces B and D the spaces C_g or C .

THEOREM 3.4. *Suppose the random integral equation (1.1) satisfies the following conditions:*

(i) *There exists a constant $Z > 0$ and a positive continuous function $g(t)$ on R_+ such that*

$$\int_0^\infty \|k_0(t, \tau; \omega)\| g(\tau) d\tau \leq Z, \quad t \in R_+;$$

(ii) *$e(t, x)$ is continuous in $t \in R_+$ uniformly in $x \in R$ such that $|e(t, 0)| \leq \gamma g(t)$ and*

$$|e(t, x) - e(t, y)| \leq \xi g(t) |x - y|$$

for $\|x\|_C, \|y\|_C \leq \rho$ and $\gamma > 0$ and $\xi > 0$ constants;

(iii) *$h(t; \omega) \in C$.*

Then there exists a unique random solution, $x(t; \omega) \in C$, such that $\|x(t; \omega)\|_C \leq \rho$, provided that $\|h(t; \omega)\|_C, \xi$, and γ are sufficiently small.

PROOF. If we show that the pair (C_g, C) is admissible with respect to the operator W given by equation (3.4), then we are through, since the other conditions are equivalent to those of Theorem 3.3.

Let $x(t; \omega) \in C_g$. Then

$$\begin{aligned} \|(Wx)(t; \omega)\|_{L_2(\Omega, \mathcal{A}, P)} &\leq \int_0^\infty \|k_0(t, \tau; \omega)x(\tau; \omega)\|_{L_2(\Omega, \mathcal{A}, P)} d\tau \\ &\leq \int_0^\infty \|k_0(t, \tau; \omega)\| \cdot \frac{\|x(\tau; \omega)\|_{L_2(\Omega, \mathcal{A}, P)}}{g(\tau)} g(\tau) d\tau \\ &\leq \sup_{t \geq 0} \left\{ \frac{\|x(t; \omega)\|_{L_2(\Omega, \mathcal{A}, P)}}{g(t)} \right\} \int_0^\infty \|k_0(t, \tau; \omega)\| g(\tau) d\tau \\ &\leq \|x(t; \omega)\|_{C_g} Z, \end{aligned}$$

by condition (i) and the definition of the norm in the space C_g . Hence, $(Wx)(t; \omega)$ is in C whenever $x(t; \omega) \in C_g$, and the pair (C_g, C) is admissible with respect to W , completing the proof.

We also obtain the following corollary.

COROLLARY 3.1. *Assume that the random integral equation (1.1) satisfies the following conditions:*

(i) $\|k_0(t, \tau; \omega)\| \leq \Lambda$, $t, \tau \in R_+$, and $\int_0^\infty g(t) dt < \infty$, where $\Lambda > 0$ is a constant;

(ii) Same as Theorem 3.4 (ii);

(iii) Same as Theorem 3.4 (iii).

Then there exists a unique random solution of (1.1) bounded on R_+ , provided that $\|h(t; \omega)\|_C$, ξ , and γ are small enough.

The proof consists of showing that (C_g, C) is admissible under (i) as in Theorem 3.4.

A comprehensive study of the stochastic integral equation (1.2) will appear in a later paper of the authors [6].

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DEPARTMENT OF STATISTICS, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VIRGINIA 24061

Current address (W. J. Padgett): Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208