BOUNDARY BEHAVIOR OF STARLIKE FUNCTIONS

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ABSTRACT. For a starlike function f , we impose a geometric condition on the image of the open unit disc by the mapping $w = zf'(z)/f(z)$ to insure that f be one-to-one on the closed unit disc. Applications are given to certain classes of univalent functions, including spiral-like functions.

1. Introduction. A function

$$(1) \quad f(z) = \sum_{n=1}^{\infty} a_n z^n$$

analytic in $|z| < 1$ is said to be starlike there if it maps $|z| < 1$ one-to-one onto a domain starshaped with respect to the origin. This is equivalent to the analytic condition that $\operatorname{Re}[zf'(z)/f(z)] > 0$ in $|z| < 1$. Moreover, if there exists α , $0 < \alpha < 1$, such that $\operatorname{Re}[zf'(z)/f(z)] > \alpha$ for $|z| < 1$, we say that f is starlike of order α .

An analytic function (1) is said to be convex if it maps $|z| < 1$ one-to-one onto a convex domain. Analytically, this is equivalent to the condition that $\operatorname{Re}[1 + zf''(z)/f'(z)] > 0$ for $|z| < 1$.

Henceforth, let D denote the open unit disc. It is clear that a bounded convex function maps D onto a domain whose boundary, $\partial f(D)$, is a rectifiable closed curve. However, if f is a bounded starlike function, then $\partial f(D)$ need not be rectifiable. For example, let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of values of θ such that $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < \cdots < \pi$, and $\lim_{n \rightarrow \infty} \theta_n = \pi$. Let l_n be the linear segment $[\frac{1}{2}e^{i\theta_n}, e^{i\theta_n}]$, and $l_0 = [\frac{1}{2}e^{i\pi}, e^{i\pi}]$. Then, if $E = D \cup \bigcup_{n=0}^{\infty} l_n$, the Riemann mapping theorem guarantees the existence of a univalent function f_1 such that $f_1(D) = E$ and $f_1(0) = 0$. Since E is a domain which is starlike with respect to the origin, it follows that f_1 is a starlike function. Furthermore, ∂E is not rectifiable.

It is known [7] that if f is convex then f is starlike of order $\frac{1}{2}$. Now the following question arises. If f is starlike of some positive order, does the boundedness of f imply that $\partial f(D)$ is rectifiable? Since

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) = \frac{\partial}{\partial \theta} \arg f(re^{i\theta}), \quad 0 < r < 1,$$

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the fact that f is starlike of order $\alpha > 0$ implies that

$$\frac{\partial}{\partial \theta} \arg f(re^{i\theta}) > \alpha, \quad 0 < r < 1.$$

Hence, one perhaps has reason to believe that $\arg f(re^{i\theta})$ increases rapidly enough, as a function of θ , so that a rectifiable boundary must result. However, this need not be the case. Simply let α satisfy $0 < \alpha < 1$ and define g_1 by

$$g_1(z) = z(f_1(z)/z)^{1-\alpha}.$$

Then g_1 is starlike of order α , but $\partial g_1(D)$ is not rectifiable. In fact, g_1 does not extend continuously to $\bar{D}$.

2. Boundary behavior. As noted above, there exist bounded starlike functions of any given order α , $0 < \alpha < 1$, which are not continuous on $\bar{D}$. If g is such a function, a theorem of Carathéodory states that $\partial g(D)$ is not a Jordan curve. But $\partial g(D)$ fails to be a Jordan curve in a nontrivial fashion; for if g is starlike of order α , $\alpha > 0$, then the finite radial limits of g are distinct. This result is contained in the following theorem.

THEOREM 1. *Let f be starlike in D , and let $\hat{f}(\theta) = \lim_{r \rightarrow 1} f(re^{i\theta})$.¹ If the intersection of the imaginary axis, I , with the closure of the set $\{w = zf'(z)/f(z) : z \in D\}$ is not a neighborhood (in I) of the origin, then $\hat{f}$ is one-to-one on the set of θ in $[-\pi, \pi)$ for which $\hat{f}(\theta)$ is finite.*

PROOF. By the familiar Herglotz representation, there exists a non-decreasing function $V(\theta)$ such that $V(\pi) - V(-\pi) = 2\pi$ and

$$(2) \quad \frac{zf'(z)}{f(z)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} dV(t).$$

Assuming $f'(0) > 0$ (without loss of generality), Keogh [2] has shown that it is possible to normalize $V(t)$ so that, for a certain fixed branch of $\arg f(z)$ and for all real θ ,

$$(3) \quad V(\theta) = \lim_{r \rightarrow 1} \arg f(re^{i\theta}).^2$$

Suppose there exist θ_1, θ_2 , $-\pi \leq \theta_1 < \theta_2 < \pi$, such that $\hat{f}(\theta_1) = \hat{f}(\theta_2) \neq \infty$. Since $V(\theta)$ is nondecreasing, $V(\theta)$ is constant on either $[\theta_1, \theta_2]$ or $[\theta_2, \theta_1 + 2\pi]$. Assuming the former, it follows from (2) that $zf'(z)/f(z)$ is

¹ By a theorem of Keogh [2], $\hat{f}(\theta)$ exists (possibly infinite) for all real values of θ .

² We define $V(\theta)$ for all θ by the condition $V(\theta + 2\pi) - V(\theta) = 2\pi$.

analytic on the arc $(e^{i\theta_1}, e^{i\theta_2})$ and has zero real part there. Again from (2),

$$(4) \quad \log \left(\frac{f(z)}{f'(0)z} \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{(1 - ze^{-it})^2} dV(t).$$

Thus, f is analytic on $(e^{i\theta_1}, e^{i\theta_2})$ and, in view of (3), the values taken by f on this arc fill a segment of some ray emanating from the origin. Taking real parts in (4), it follows by Rolle's theorem that there exists $\theta_0 \in (\theta_1, \theta_2)$ such that $(\partial/\partial\theta)\log|\hat{f}(\theta)|=0$ at $\theta=\theta_0$. Hence, $\text{Im}(zf'(z)/f(z))=0$ at $z=e^{i\theta_0}$; and so, there exists an open interval of the imaginary axis which contains the origin and belongs to the closure of $\{w=zf'(z)/f(z): z \in D\}$. The proof is complete.

One may ask whether the converse is true. That is, if the intersection of the imaginary axis, I , with the closure of $\{w=zf'(z)/f(z): z \in D\}$ is a neighborhood (in I) of the origin, does $\hat{f}$ fail to be one-to-one on the set of θ in $[-\pi, \pi]$ for which $\hat{f}(\theta)$ is finite? The answer is no, in general. In fact, let G be any simply connected domain (containing the point 1) lying in the right half plane, having a prime end that is a neighborhood (in I) of the origin; and let g be a Riemann mapping function from D onto G , $g(0)=1$. If f is any solution of the differential equation $zf'(z)=g(z)f(z)$,³ f is starlike in D and $\hat{f}$ is one-to-one on the set of θ in $[-\pi, \pi]$ for which $\hat{f}(\theta)$ is finite. This follows by the same argument used in Theorem 1.

Theorem 1 and a well-known theorem of Carathéodory yield the following corollary.

COROLLARY. *Under the hypotheses of Theorem 1, the following are equivalent:*

- (a) f is continuous on $\bar{D}$.
- (b) $\partial f(D)$ is a Jordan curve.

3. Examples. The following classes of functions satisfy the conditions of Theorem 1.

A. Starlike functions of order α , $0 < \alpha < 1$.

B. Strongly starlike functions of order α , $0 < \alpha < 1$.

By definition, the starlike function f is said to be strongly starlike of order α ($0 < \alpha < 1$) if

$$|\arg(zf'(z)/f(z))| \leq \alpha(\pi/2), \quad z \in D.$$

This class of functions has recently been examined by Brannan and Kirwan [1]. They show that (a) and (b) of the corollary hold for this class. In fact, if f is strongly starlike of order α ($0 < \alpha < 1$), then $f(D)$ has a rectifiable boundary curve.

³ The function $z \exp \int_0^z (g(w)-1)w^{-1} dw$ is easily seen to be a solution.

C. We now consider a class of functions which satisfies a "rotated" form of Theorem 1.

DEFINITION. If f be analytic in D , $f'(0) \neq 0$, then f is said to be α -spiral-like, $-\pi/2 < \alpha < \pi/2$, if

$$(5) \quad \operatorname{Re}(e^{i\alpha}zf'(z)/f(z)) > 0, \quad z \in D.$$

If $\alpha=0$ we have the class of starlike functions. Špaček [6] has shown that if an analytic function f satisfies (5), then f must be univalent in D . Further properties of α -spiral-like functions can be found in [3]. Note that (5) is equivalent to the statement that the image of D under the mapping $zf'(z)/f(z)$ should lie to the right of the line $w = te^{i(\pi/2-\alpha)}$, t real.

If f is α -spiral-like, the function g , defined by

$$g(z) = z(f(z)/z)^{1+i \tan \alpha}, \quad z \in D,$$

is starlike in D . It follows that f has radial limits (possibly infinite) for all values of θ . Furthermore, Theorem 1 and the corollary hold for α -spiral-like functions, if one replaces " I " by "line $w = te^{i(\pi/2-\alpha)}$, t real."

The following theorem illustrates the geometric distinction between spiral-like and starlike functions.

THEOREM 2. Suppose f is both α and β -spiral-like, $-\pi/2 < \alpha < \beta < \pi/2$. Then f is γ -spiral-like, $\forall \gamma$, $\alpha \leq \gamma \leq \beta$; and f maps D onto a domain whose boundary is a rectifiable Jordan curve.

PROOF. The fact that $\operatorname{Re}(e^{i\gamma}zf'(z)/f(z)) > 0$ for $z \in D$ follows from the geometry of the situation. Furthermore, there exists ρ , $0 < \rho < 1$, such that

$$(6) \quad \frac{zf'(z)}{f(z)} < [1 - ze^{i(\pi-(\alpha+\beta))/2}]^\rho [1 - ze^{-i(\pi-(\alpha+\beta))/2}]^{-\rho}.$$

By an inequality of Littlewood [4], we have

$$\int_{-\pi}^{\pi} \left| \frac{zf'(z)}{f(z)} \right| d\theta \leq 2^\rho \int_{-\pi}^{\pi} |1 - re^{i\theta}|^{-\rho} d\theta.$$

Since $0 < \rho < 1$, $\int_{-\pi}^{\pi} |zf'(z)/f(z)| d\theta$ remains bounded as $r \rightarrow 1$, and so $zf'(z)/f(z) \in H$. It follows also from (6) that f is bounded; thus $f' \in H$. By a theorem of Seidel [5], $\partial f(D)$ is rectifiable. Also, (6) shows that the hypotheses of the spiral-like formulation of Theorem 1 hold; hence, part (a) of the corollary holds (since $f' \in H$). The proof is complete.

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* All powers are meant as principal values.

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