ON THE LEBEDEV-MILIN INEQUALITY¹

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ABSTRACT. A generalization is given of an inequality of Lebedev-Milin involving the coefficients of a power series f and those of $\exp(f)$. It is proved that the exponential function can be replaced by certain functions belonging to the family of functions whose Taylor expansions at the origin have nonnegative coefficients.

In 1965 Lebedev and Milin [3] introduced a simple inequality which proved extremely useful in geometric function theory. We mention only three results where their inequality was used: (a) Milin [4] proved for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ (S the class of normalized univalent functions in $|z| < 1$) that $|a_n| < (1.243)^n$. (b) Il'ina [2] proved for $f \in S$ that $||a_{n+1}| - |a_n|| < 4.26$. (c) Aharonov [1] and Nehari [5] proved for $F(z) = \sum_{n=1}^{\infty} b_n z^n \in \text{B.E.}$ (B.E. the Bieberbach-Eilenberg class of functions $F(z)$ regular in $|z| < 1$ and $F(z)F(\zeta) \neq 1$ for any $|z|, |\zeta| < 1$) that $\sum_{n=1}^{\infty} |b_n|^2 \leq 1$.

We will give here a generalization of their inequality using the same method of proof as in [3].

THEOREM. Suppose that $g(t) = 1 + \sum_{k=1}^{\infty} p_k t^k$, $0 \leq p_k$, converges for $|t| < R$, $0 < R \leq \infty$, and that the transformed inverse function $g^{-1}(1/(1-s)) = \sum_{k=1}^{\infty} q_k s^k$ has positive coefficients. Let $\{A_k\}_{k=1}^{\infty}$ be an arbitrary sequence of complex numbers satisfying $\sum_{k=1}^{\infty} |A_k|^2/q_k < R$. Then from

$$(1) \quad g\left(\sum_{k=1}^{\infty} A_k z^k\right) = \sum_{k=0}^{\infty} D_k z^k$$

follows

$$(2) \quad \sum_{k=0}^{\infty} |D_k|^2 \leq g\left(\sum_{k=1}^{\infty} |A_k|^2/q_k\right),$$

with equality if and only if $A_k = q_k \rho^k \eta^k$, where $0 \leq \rho < [1 - 1/g(R)]^{1/2}$ and $|\eta| = 1$.

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PROOF. Denoting $a_k = |A_k|/q_k$ we set

$$g\left(\sum_{k=1}^{\infty} a_k q_k z^k\right) = \sum_{k=0}^{\infty} B_k z^k$$

and

$$g\left(\sum_{k=1}^{\infty} a_k^2 q_k z^k\right) = \sum_{k=0}^{\infty} C_k z^k.$$

Since $0 \leq p_k$, we have $|D_k| \leq B_k$. Comparing coefficients,

$$\begin{aligned} B_0 &= 1, & B_2 &= p_1 q_2 a_2 + p_2 q_1^2 a_1^2, \\ B_1 &= p_1 q_1 a_1, & B_3 &= p_1 q_3 a_3 + 2p_2 q_1 q_2 a_1 a_2 + p_3 q_1^3 a_1^3, \end{aligned}$$

and in general $B_k = \sum_{\mu} \alpha_{\mu,k} U_{\mu,k}$ where $\alpha_{\mu,k}$ are nonnegative coefficients depending on p_i and q_i ($i=1, \dots, k$) and $U_{\mu,k}$ are products of powers of a_i ($i=1, \dots, k$) of degree k . Clearly, $C_k = \sum_{\mu} \alpha_{\mu,k} U_{\mu,k}^2$. Also, if we set all $a_k=1$, we get

$$g\left(\sum_{k=1}^{\infty} q_k z^k\right) = g(g^{-1}(1/(1-z))) = \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} \sum_{\mu} \alpha_{\mu,k} z^k$$

and hence $\sum_{\mu} \alpha_{\mu,k} = 1$ for $k=1, 2, \dots$.

By Cauchy's inequality

$$B_k^2 = \left\{ \sum_{\mu} (\alpha_{\mu,k})^{1/2} ((\alpha_{\mu,k})^{1/2} U_{\mu,k}) \right\}^2 \leq \sum_{\mu} \alpha_{\mu,k} \cdot \sum_{\mu} \alpha_{\mu,k} U_{\mu,k}^2 = C_k$$

and so $\sum_{k=0}^{\infty} |D_k|^2 \leq \sum_{k=0}^{\infty} B_k^2 \leq \sum_{k=0}^{\infty} C_k = g(\sum_{k=1}^{\infty} |A_k|^2/q_k)$ which proves (2).

The conditions for equality are easily verified by observing that we must have $U_{\mu,k} = c_k$ and recursively $U_{\mu,k} = \rho^k$ ($\rho = c_1$). Hence $|A_k| = q_k a_k = q_k \rho^k$ and finally $|D_k| = B_k$ implies $A_k = q_k \rho^k \eta^k$ where $|\eta| = 1$.

Note that the special case $p_k = 1/k!$ and $q_k = 1/k$ gives the classic Lebedev-Milin inequality:

$$(3) \quad \exp \sum_{k=1}^{\infty} A_k z^k = \sum_{k=0}^{\infty} D_k z^k$$

implies

$$(4) \quad \sum_{k=0}^{\infty} |D_k|^2 \leq \exp \sum_{k=1}^{\infty} k |A_k|^2.$$

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