EXTENSIONS OF LEFT UNIFORMLY CONTINUOUS FUNCTIONS ON A TOPOLOGICAL SEMIGROUP

SAMUEL J. WILEY¹

ABSTRACT. For any topological semigroup S with separately continuous operation, let $C(S)$ denote the set of all bounded continuous real valued functions on S with the supremum norm and let $LUC(S)$ denote the set of all f in $C(S)$ such that whenever $\{s(\gamma)\}$ is a net in S which converges to some s in S , then $\sup\{|f(s(\gamma)t) - f(st)| : t \in S\}$ converges to 0. In this paper we prove that if S is an abelian subsemigroup of a compact topological group and $f \in LUC(S)$, then there is an $F \in LUC(G)$ where $F(s) = f(s)$ for all $s \in S$. We also show whenever there is an extension of the type indicated above, there is a norm preserving extension.

1. Introduction. By a *topological semigroup* we mean a semigroup with a Hausdorff topology in which the product st is separately continuous. If X is a topological space, $C(X)$ indicates the algebra of all bounded continuous real valued functions on X with the supremum norm. If S is a topological semigroup, $l_s: C(S) \rightarrow C(S)$ is defined by $l_s f(t) = f(st)$ and $r_s: C(S) \rightarrow C(S)$ is defined by $r_s f(t) = f(ts)$. Let S be a topological semigroup. $f \in C(S)$ is said to be *left uniformly continuous* or $f \in LUC(S)$ when the function $\theta: S \rightarrow C(S)$ defined by $\theta(s) = l_s f$ is continuous on S . The space $LUC(S)$ is a closed subalgebra of $C(S)$ which contains the constants [8]. Mitchell [7] considered topological semigroups T and subsemigroups S . He showed if S is dense in T and $LUC(S)$ has a left invariant mean, so has $LUC(T)$. In the next section we will show that if S is an abelian subsemigroup of a compact topological group, then every $f \in LUC(S)$ has an extension to an $F \in LUC(T)$.

If $(X, \mathcal{U})$ is a uniform space, then $UC(X)$ denotes the set of all bounded real valued functions on X which are uniformly continuous with respect to $\mathcal{U}$. In the following proposition the equivalence of (a) and (b) follows from

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Kelley [5, p. 86], (b) implies (c) is found in Granirer and Lau [2, Lemma 3], the rest comes directly from the definitions.

PROPOSITION 1. *Let S be a topological semigroup and $f \in C(S)$. The following are equivalent.*

- (a) $f \in \text{LUC}(S)$.
 - (b) *For every $s \in S$, if $\{s(\gamma)\}$ is a net converging to s , then $l_{s(\gamma)}f$ converges to $l_s f$ uniformly.*
 - (c) $\{r_t f : t \in S\}$ is an equicontinuous family of functions on S .
- If S is a topological group, then (a), (b) and (c) are equivalent to (d).*
- (d) $f \in \text{UC}(S)$ where the uniformity on S is the usual right uniformity on a topological group [1, p. 243].

2. Extension theorems. It will be convenient to have the following definition. Let S, T be topological semigroups, S a subsemigroup of T . Then we will say the pair (S, T) has *property P*, if for every $f \in \text{LUC}(S)$ there is an $F \in \text{LUC}(T)$ such that $F|_S = f$. $F|_S$ indicates the restriction of F to S .

PROPOSITION 2. *Let G be a topological group with the right uniformity and S a subsemigroup of G with the restriction uniformity. The following are equivalent.*

- (a) *The pair (S, G) has property P.*
- (b) $\text{LUC}(S) \subset \text{UC}(S)$.
- (c) $\text{LUC}(S) = \text{UC}(S)$.

PROOF. (a) $\rightarrow$ (b) If $f \in \text{LUC}(S)$, then $f = F|_S$ where $F \in \text{LUC}(G) = \text{UC}(G)$. Hence $f \in \text{UC}(G)|_S \subset \text{UC}(S)$.

(b) $\rightarrow$ (c) If $f \in \text{UC}(S)$, by Katetov [4, Theorem 3], $f \in \text{UC}(G)|_S = \text{LUC}(G)|_S$. Hence, by Lau [6, Proposition 1.3.1], $f \in \text{LUC}(S)$.

(c) $\rightarrow$ (a) If $f \in \text{LUC}(S) = \text{UC}(S)$, by Katetov [4, Theorem 3] there is an $F \in \text{UC}(G) = \text{LUC}(G)$ such that $F|_S = f$.

The following theorem is an immediate consequence of Propositions 1 and 2.

THEOREM 3. *If G is a topological group and G' is a subgroup of G , then the pair (G', G) has property P.*

The following example shows that we do not always have the extension property from a subsemigroup of a topological group to the group.

EXAMPLE. Let G be the real numbers under addition, S be the positive real numbers and $f(x) = \sin(1/x)$. f is not uniformly continuous on S since it does not have a continuous extension to $\text{cl}(S)$. But $f \in \text{LUC}(S)$. So the pair (S, G) does not have property P. We can see $f \in \text{LUC}(S)$ since if

$\{s(n)\}$ is a net in S converging to $s \in S$, then

$$\begin{aligned} |l_{s(n)}f(x) - l_sf(x)| &= \left| \sin \frac{1}{s(n)+x} - \sin \frac{1}{s+x} \right| \\ &= \left| 2 \sin \frac{1}{2} \left(\frac{1}{s(n)+x} - \frac{1}{s+x} \right) \cos \frac{1}{2} \left(\frac{1}{s(n)+x} + \frac{1}{s+x} \right) \right| \\ &\leq 2 \left| \sin \frac{1}{2} \frac{s-s(n)}{(s(n)+x)(s+x)} \right|. \end{aligned}$$

But s is positive, so for sufficiently large n , $s(n) > \delta$ and $s > \delta$ where $\delta > 0$. Since $x > 0$, $s(n)+x > \delta$ and $s+x > \delta$. So

$$\left| \frac{s-s(n)}{(s(n)+x)(s+x)} \right| < \left| \frac{s-s(n)}{\delta^2} \right| \quad \text{for all } x \in S.$$

Since the sine function is continuous and $(s-s(n))/\delta^2$ converges to 0, there is an n_0 such that for all $n \geq n_0$,

$$2 \left| \sin \frac{1}{2} \frac{s-s(n)}{(s(n)+x)(s+x)} \right| < \varepsilon \quad \text{for all } x \in S.$$

So $f \in \text{LUC}(S)$.

Mitchell [7, pp. 640–641] has given an example of a compact topological semigroup T and a dense subsemigroup S of T where the pair (S, T) does not have property P . We will now show that we have the desired extension whenever T is a compact topological group and S is abelian.

THEOREM 4. *If G is a compact topological group and S is an abelian subsemigroup of G , then the pair (S, G) has property P .*

PROOF. (i) Since $\text{cl}(S)$ is a compact cancellation semigroup, it is a topological group [3, Theorem 9.16]. Hence by Theorem 3 we may assume $\text{cl}(S) = G$.

(ii) Let $f \in \text{LUC}(S)$, $x(\alpha)$ be a net in S which converges to e and $a \in S$. Then $x(\alpha)a$ converges to a . Hence $\lim_\alpha f(x(\alpha)ay) = f(ay)$ uniformly for all y in S .

(iii) Now since G is compact, there is a unique uniformity for G . This consists of all neighborhoods of the diagonal Δ [5, p. 197]. Let $A_\varepsilon = \{(x, y) \in S \times S : |f(x) - f(y)| \geq \varepsilon\}$. Then f is uniformly continuous on S iff for every $\varepsilon > 0$, $\text{cl}(A_\varepsilon) \cap \Delta = \emptyset$. Suppose there is an $f \in \text{LUC}(S)$ where $f \notin \text{UC}(S)$. Then there is an $\varepsilon > 0$ such that $\text{cl}(A_\varepsilon) \cap \Delta \neq \emptyset$. Hence there is a net $(x(\alpha), y(\alpha))$ in A_ε which converges to $(t, t) \in \Delta$. Now let $\{t(\beta)\}$ be a net in S which converges to t^{-1} . Then $\lim_{\alpha, \beta} (x(\alpha)t(\beta)) = e = \lim_{\alpha, \beta} (y(\alpha)t(\beta))$. Then by part (ii) above given $\varepsilon > 0$ there are α and β such that

$$|f(x(\alpha)t(\beta)s) - f(s)| < \varepsilon/2 \quad \text{and} \quad |f(y(\alpha)t(\beta)s) - f(s)| < \varepsilon/2$$

for all $s \in S$. But $x(\alpha) \in S$ and $y(\alpha) \in S$ hence

$$\begin{aligned} |f(x(\alpha)) - f(y(\alpha))| &\leq |f(x(\alpha)) - f(y(\alpha)t(\beta)x(\alpha))| \\ &\quad + |f(x(\alpha)t(\beta)y(\alpha)) - f(y(\alpha))| < \varepsilon. \end{aligned}$$

This contradicts that $(x(\alpha), y(\alpha)) \in A_\varepsilon$.

COROLLARY 5. *If G is a topological group, S is an abelian subsemigroup of G where $\text{cl}(S)$ is compact, then the pair (S, G) has property P .*

PROOF. $\text{cl}(S)$ is a compact topological group [3, Theorem 9.16]. So we can make the extension from S to $\text{cl}(S)$. Then by Theorem 3 we can make the extension from $\text{cl}(S)$ to G .

Finally we will show that whenever we have an extension, we have a norm preserving extension.

PROPOSITION 6. *If S is a subsemigroup of a topological semigroup T and the pair (S, T) has property P , then given $f \in \text{LUC}(S)$ there is a $g \in \text{LUC}(T)$ such that $g|_S = f$ and $\|g\| = \|f\|$.*

PROOF. Let $F \in \text{LUC}(T)$ be an extension of $f \in \text{LUC}(S)$. If $g = (F \wedge \|f\|) \vee (-\|f\|)$ where $\|f\|$ and $-\|f\|$ are constant functions, then $g \in \text{LUC}(T)$ [8, Lemma 1.1] and is the desired norm preserving extension of f .

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DEPARTMENT OF MATHEMATICS, LA SALLE COLLEGE, PHILADELPHIA, PENNSYLVANIA 19141