MAPS BETWEEN TOPOLOGICAL GROUPS THAT ARE HOMOTOPIC TO HOMOMORPHISMS

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ABSTRACT. Let G be a compact connected group and let H be a locally compact abelian group. Denote by $C_e(G, H)$ the space of all identity preserving continuous functions from G to H with the compact-open topology, and denote by $\text{Hom}(G, H)$ the space of all homomorphisms in $C_e(G, H)$. We prove that $C_e(G, H)$ is isomorphic to $V \times \text{Hom}(G, H)$, where V is a topological vector space. This is used to prove that every element of $C_e(G, H)$ is homotopic to precisely one element of $\text{Hom}(G, H)$. We also prove that the fundamental group of H is isomorphic to $\text{Hom}(K, H)$, K being the circle group, that $\pi_n(H) = 0$ for $n \geq 2$, and that a compact connected abelian group admits essentially only one H -space structure.

Introduction. Plunkett [6] showed that every continuous function from a compact, connected, abelian, and 2° countable group into the circle group is homotopic to exactly one continuous homomorphism. Corollary 2 is a generalization of this. Enochs [3] proved Corollary 3 in the case where the group is compact.

Notation. The phrase "continuous function" will be shortened to "map", and "space" will mean "topological space". If X and Y are spaces, then $C(X, Y)$ denotes the space of all maps from X to Y with the compact-open topology. If $x \in X$ then $\pi_n(X, x)$ is the n th homotopy group of X with base point x . Topological groups will always be assumed Hausdorff, and abelian groups will be written additively. If A and B are topological groups, then $C_e(A, B)$ is $\{f \in C(A, B) \mid f(1) = 0\}$, where 1 and 0 are the respective identities, and $\text{Hom}(A, B)$ is $\{f \in C_e(A, B) \mid f \text{ is a homomorphism}\}$. One can verify that, if A is locally compact and B is abelian, then $C_e(A, B)$ and $\text{Hom}(A, B)$ are topological groups with the relative topology and pointwise operations.

The group of real members and the group of integers are denoted by R and Z , respectively. We set $K = R/Z$, and $p: R \rightarrow K$ is the natural map.

Received by the editors June 18, 1971.

AMS 1970 subject classifications. Primary 22C05, 54F60, 55A05, 55E05; Secondary 55D45.

Key words and phrases. Compact-open topology, homotopy group, homotopic maps, H -space, normalized Haar integral, topological vector space.

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If H is a locally compact abelian group, then H^* denotes $\text{Hom}(H, K)$, and the canonical isomorphism T from H onto H^{**} is given by $T(x)(\chi) = \chi(x)$ [7, p. 273]. The map $p': \text{Hom}(H^*, R) \rightarrow H$ is defined by $p'(h) = T^{-1}(p \circ h)$.

1. Statement of Theorems 1 and 2. If G and H are locally compact groups, and if H is abelian, then the functions

$$F: C_e(G, R) \times \text{Hom}(G, K) \rightarrow C_e(G, K),$$

$$F': C_e(G, \text{Hom}(H^*, R)) \times \text{Hom}(G, H) \rightarrow C_e(G, H)$$

are defined by $F(f, h) = (p \circ f) + h$ and $F'(f, h) = (p' \circ f) + h$.

THEOREM 1. *If G is a compact connected topological group then the function F is an algebraic and topological isomorphism of $C_e(G, R) \times \text{Hom}(G, K)$ onto $C_e(G, K)$.*

THEOREM 2. *If G is a compact connected topological group, and if H is a locally compact abelian group, then the function F' is an algebraic and topological isomorphism of $C_e(G, \text{Hom}(H^*, R)) \times \text{Hom}(G, H)$ onto $C_e(G, H)$.*

Note that $C_e(G, \text{Hom}(H^*, R))$ is a topological vector space because $\text{Hom}(H^*, R)$ is one. On the other hand, $\text{Hom}(G, H)$ is totally disconnected if G and H are as in Theorem 2, and $\text{Hom}(G, H)$ is discrete if $H = K$. Theorem 1 is a special case of Theorem 2, for when $H = K$ we have that $\text{Hom}(H^*, R)$ is isomorphic to R , and that p' coincides with p . Nevertheless, Theorem 2 is an easy consequence of Theorem 1.

2. Lemmas for Theorems 1 and 2.

LEMMA 1. *Let X , Y and Z be locally compact Hausdorff spaces. Then*

- (i) *the function from $C(X, Y) \times C(Y, Z)$ to $C(X, Z)$ taking (f, g) to $g \circ f$ is continuous,*
- (ii) *there is a homeomorphism D from $C(X \times Y, Z)$ onto $C(X, C(Y, Z))$ given by $((D(f))(x))(y) = f(x, y)$,*
- (iii) *there is a homeomorphism E from $C(X, C(Y, Z))$ onto $C(Y, C(X, Z))$ given by $((E(f))(y))(x) = (f(x))(y)$.*

PROOF. Parts (i) and (ii) can be found in [2, pp. 259, 265]. Part (iii) follows from part (ii) because E can be factored as $C(X, C(Y, Z)) \rightarrow C(X \times Y, Z) \rightarrow C(Y \times X, Z) \rightarrow C(Y, C(X, Z))$ in the natural way.

If G is a locally compact group then the function $I: C_e(G, R) \rightarrow C_e(G, K)$ is defined by $I(f) = p \circ f$.

LEMMA 2. *If G is a compact connected topological group then the function $I: C_e(G, R) \rightarrow C_e(G, K)$ is continuous, open, one-to-one, and a homomorphism.*

PROOF. The function I is continuous by Lemma 1(i), and it is a homomorphism because p is one. To prove that I is one-to-one, let $f \in C_e(G, R)$ and suppose that $I(f) = 0$. Then $f(G) \subset \text{Ker } p = Z$, and since G is connected and Z is discrete we obtain $f(G) = 0$, proving $f = 0$.

To prove that I is open, it suffices to show that $I(U)$ is open if U is a basic open set about $0 \in C_e(G, R)$. We can assume that

$$U = \{f \in C_e(G, R) \mid f(G) \subset V\},$$

where V is an open neighborhood of $0 \in R$ and the diameter of V is less than 1. If we set $W = \{g \in C_e(G, K) \mid g(G) \subset p(V)\}$, then W is open in $C_e(G, K)$ because $p(V)$ is open in K . We will finish the proof by showing that $I(U) = W$.

It is clear that $I(U) \subset W$. If $g \in W$ then, since $g(G) \neq K$, g can be factored as $g = p \circ f$, where $f \in C_e(G, R)$. Then $f(G) \subset p^{-1}(p(V))$ and the identity component of $p^{-1}(p(V))$ is contained in V . Since G is connected, it follows that $f(G) \subset V$. Hence $g = I(f)$ where $f \in U$.

3. Proof of Theorem 1. Let G be a compact connected group. Our first step is to show that F is onto.

Let $f \in C_e(G, K)$. We will express f as $f = (p \circ g) + h$, where $g \in C_e(G, R)$ and $h \in \text{Hom}(G, K)$. For each $a \in G$, let $f_a \in C_e(G, K)$ be defined by $f_a(b) = f(a) + f(b) - f(ab)$. The function from G to $C_e(G, K)$ taking a to f_a is continuous by Lemma 1(ii), for the corresponding function from $G \times G$ to K is continuous. By Lemma 2, $I(C_e(G, R))$ is an open subgroup of $C_e(G, K)$, and hence $I(C_e(G, R))$ is also closed in $C_e(G, K)$. Since G is connected, $f_1 = 0$, and the function $a \rightarrow f_a$ is continuous, we conclude that $f_a \in I(C_e(G, R))$ for every $a \in G$. Let $f_a = I(g_a)$, where $g_a \in C_e(G, R)$. Then the function from G to $C_e(G, R)$ taking a to g_a is continuous because I is open and one-to-one (Lemma 2).

Define $g \in C_e(G, R)$ by $g(a) = \int g_a(x) dx$, where the integral used is the normalized Haar integral on G [4, pp. 9-13]. Next, we establish the formula

$$(1) \quad g_a(b) = g_a(bx) + g_b(x) - g_{ab}(x),$$

valid for all $a, b, x \in G$. Fix $a, b \in G$ and define $k \in C_e(G, R)$ by $k(x) = g_a(bx) + g_b(x) - g_{ab}(x) - g_a(b)$. Then

$$(p \circ k)(x) = f_a(bx) + f_b(x) - f_{ab}(x) - f_a(b) = 0.$$

Hence $I(k) = 0$, and by Lemma 2 we have $k = 0$, proving (1).

If $a, b \in G$ then, using the translation invariance of the integral and formula (1), we obtain

$$\begin{aligned}
 g(a) + g(b) - g(ab) &= \int g_a(x) dx + \int g_b(x) dx - \int g_{ab}(x) dx \\
 &= \int g_a(bx) dx + \int g_b(x) dx - \int g_{ab}(x) dx \\
 (2) \qquad &= \int [g_a(bx) + g_b(x) - g_{ab}(x)] dx \\
 &= \int g_a(b) dx = g_a(b).
 \end{aligned}$$

Applying p to (2), we obtain

$$(3) \quad (p \circ g)(a) + (p \circ g)(b) - (p \circ g)(ab) = f_a(b) = f(a) + f(b) - f(ab).$$

Let $h = f - (p \circ g)$. Then (3) shows that $h \in \text{Hom}(G, K)$. Hence $f = (p \circ g) + h$, where $g \in C_e(G, R)$ and $h \in \text{Hom}(G, K)$. This proves that F is onto.

The function F is clearly a homomorphism. Our next step is to show that F is one-to-one. Suppose $(p \circ f) + h = 0$, where $f \in C_e(G, R)$ and $h \in \text{Hom}(G, K)$. Then $p \circ f \in \text{Hom}(G, K)$. To show that $f \in \text{Hom}(G, R)$, fix $a \in G$ and define $g \in C_e(G, R)$ by $g(x) = f(a) + f(x) - f(ax)$. Then, since $(p \circ f) \in \text{Hom}(G, K)$, we obtain

$$(p \circ g)(x) = (p \circ f)(a) + (p \circ f)(x) - (p \circ f)(ax) = 0.$$

Hence $I(g) = 0$, and this implies $g = 0$ by Lemma 2. This shows that f is a homomorphism. Since G is compact, $f(G)$ is a compact subgroup of R , forcing $f(G) = 0$ and $f = 0$. Therefore h is also zero and this proves that $\text{Ker } F = 0$.

The function F is continuous by Lemma 1(i). We finish the proof of Theorem 1 by showing that F is an open map. First, we must show that $\text{Hom}(G, K)$ is discrete. If V is a neighborhood of $0 \in K$ such that $V \neq K$, and if we set $U = \{f \in \text{Hom}(G, K) \mid f(G) \subset V\}$, then U is a neighborhood of $0 \in \text{Hom}(G, K)$. Since G is connected, $U = \{0\}$. This proves that $\text{Hom}(G, K)$ is discrete and hence that $C_e(G, R) \times \{0\}$ is an open subset of $C_e(G, R) \times \text{Hom}(G, K)$. Then F is open because the restriction of F to $C_e(G, R) \times \{0\}$ is open by Lemma 2. This finishes the proof of Theorem 1.

4. Proof of Theorem 2. Let G be a compact connected group, and let H be a locally compact abelian group. Applying the functor $\text{Hom}(H^*, -)$ to the isomorphism $F: C_e(G, R) \times \text{Hom}(G, K) \rightarrow C_e(G, K)$, we obtain an isomorphism

$$\begin{aligned}
 (4) \qquad &\text{Hom}(H^*, C_e(G, R)) \times \text{Hom}(H^*, \text{Hom}(G, K)) \\
 &\rightarrow \text{Hom}(H^*, C_e(G, K)).
 \end{aligned}$$

When we apply the isomorphism of Lemma 1(iii) to each group in (4) we obtain an isomorphism

$$(5) \quad \begin{aligned} C_e(G, \operatorname{Hom}(H^*, R)) \times \operatorname{Hom}(G, \operatorname{Hom}(H^*, K)) \\ \rightarrow C_e(G, \operatorname{Hom}(H^*, K)). \end{aligned}$$

Finally, applying the isomorphism $T: H \rightarrow H^{**}$ to (5), we obtain an isomorphism

$$(6) \quad C_e(G, \operatorname{Hom}(H^*, R)) \times \operatorname{Hom}(G, H) \rightarrow C_e(G, H).$$

The isomorphism in (6) is F' .

5. Applications of Theorem 2.

COROLLARY 1. *Let G be a compact connected topological group, and let H be a locally compact abelian topological group. Then $\operatorname{Hom}(G, H)$ is a strong deformation retract of $C_e(G, H)$.*

PROOF. This follows from Theorem 2 because $C_e(G, \operatorname{Hom}(H^*, R))$ is contractible.

COROLLARY 2. *Let G be a compact connected topological group, and let H be a locally compact abelian topological group. Then every $f \in C_e(G, H)$ is homotopic to exactly one $h \in \operatorname{Hom}(G, H)$, and the homotopy can be chosen to preserve the identity.*

PROOF. By Theorem 2, $f = (p' \circ g) + h$, where $g \in C_e(G, V)$ and V is a topological vector space. Since g is homotopic to 0 by a homotopy that preserves the identity, f is homotopic to h in the same way.

To prove that f is homotopic to at most one homomorphism, suppose that f is homotopic to h_1 and h_2 , where $h_1, h_2 \in \operatorname{Hom}(G, H)$. Then h_1 is homotopic to h_2 and hence, setting $h = h_1 - h_2$, h is homotopic to 0. We will show that $\chi \circ h = 0$ for every $\chi \in H^*$. This will prove that $h = 0$ and hence that $h_1 = h_2$. If $\chi \in H^*$ then $\chi \circ h$ is an element of $\operatorname{Hom}(G, K)$ homotopic to 0. Since $p: R \rightarrow K$ is a covering projection [8, Chapter 2], $\chi \circ h = p \circ g$ for some $g \in C_e(G, R)$. Then $\chi \circ h$ can be expressed as $\chi \circ h = F(g, 0)$ and $\chi \circ h = F(0, \chi \circ h)$. Since F is one-to-one by Theorem 1, we obtain $\chi \circ h = 0$.

One can see from Corollary 2 how essential our conditions on G and H are. The corollary is clearly false if $G = R$ and $H = K$, and in the case where G is the discrete two point group and $H = K$. If S^3 denotes the group of quaternions of modulus 1, then $\operatorname{Hom}(K, S^3)$ is nonzero but every element of $C_e(K, S^3)$ is homotopic to 0, showing that commutativity of H is necessary.

COROLLARY 3. *Let H be a locally compact abelian group. Then $\pi_1(H, 0)$ is isomorphic to $\operatorname{Hom}(K, H)$, and $\pi_n(H, 0) = 0$ for $n \geq 2$.*

PROOF. We will consider each element of $\pi_1(H, 0)$ as an equivalence class of elements of $C_e(K, H)$. Let $J: \text{Hom}(K, H) \rightarrow \pi_1(H, 0)$ be the function that takes each element of $\text{Hom}(K, H)$ to its equivalence class. Then J is a homomorphism by [8, p. 51], and J is a bijection by Corollary 2.

Let $n \geq 2$ and let W be the space of null homotopic elements of $C_e(K, H)$. By [5, p. 109], $\pi_n(H, 0)$ is isomorphic to $\pi_{n-1}(W, 0)$. By the proof of Corollary 2, W is the image under F' of $C_e(K, \text{Hom}(H^*, R)) \times \{0\}$, and hence W is contractible. Therefore $\pi_{n-1}(W, 0) = 0$, forcing $\pi_n(H, 0) = 0$.

Our last corollary deals with H -spaces. If X is a space and $n: X \times X \rightarrow X$ is a continuous function, then $e \in X$ is called a homotopy identity for n iff the maps $k_1: X \rightarrow X$ and $k_2: X \rightarrow X$, defined by $k_1(x) = n(e, x)$ and $k_2(x) = n(x, e)$, are each homotopic to the identity on X . If n has a homotopy identity then (X, n) is called an H -space [8, p. 34]. Corollary 4 is related to a result by Robert Brown [1].

COROLLARY 4. *Let G be a compact connected abelian group with identity e and group multiplication $m: G \times G \rightarrow G$. Suppose (G, n) is an H -space with e as homotopy identity and $n(e, e) = e$. Then m and n are homotopic.*

PROOF. Let $i_1: G \rightarrow G \times G$ and $i_2: G \rightarrow G \times G$ be defined by $i_1(x) = (e, x)$ and $i_2(x) = (x, e)$. We will write $m(x, y) = xy$. By Corollary 2, n is homotopic to some $h \in \text{Hom}(G \times G, G)$. For $j \in \{1, 2\}$, $h \circ i_j$ is a homomorphism homotopic to $n \circ i_j = k_j$. Since k_j is homotopic to the identity on G , Corollary 2 shows that $h \circ i_j$ is the identity on G . This means that $h(e, x) = x = h(x, e)$ for all $x \in G$. Hence for any $a, b \in G$ we have $h(a, b) = h(ae, eb) = h(a, e)h(e, b) = ab = m(a, b)$. This proves that $h = m$, and hence that n is homotopic to m .

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