APPLICATIONS OF NILMANIFOLD THEORY TO DIOPHANTINE APPROXIMATIONS

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ABSTRACT. We indicate one method for applying harmonic analysis of nilmanifolds to problems of uniform distribution mod 1 in $\mathbf{R}$.

1. Introduction. We are going to illustrate a technique for using the results of [1] and [3] to prove theorems on the uniform distribution of sequences mod 1 in $\mathbf{R}$. Two results will be proved: one classical, the other new. The classical result is H. Weyl's theorem (see [5]) that whenever P is a polynomial over $\mathbf{R}$ at least one of whose coefficients (other than the constant term) is irrational, the sequence $P(1), P(2), \dots$ is uniformly distributed mod 1. Our proof of Weyl's theorem is closely related to a proof given by F. Hahn in [4]. Hahn was dealing with affine maps from a torus to itself, and for that reason his methods do not extend in an obvious way to get our new result, which is that whenever τ is an irrational number and n is an integer greater than 1, the sequence² $a_r = (\tau n!)^{-1}(\{\tau r\}^n + (-1)^{n+1}[\tau r]^n)$ is uniformly distributed mod 1. Our proof of this result uses in an essential way certain affine maps of a nilmanifold which is visibly not a torus.

Our computations all deal with functions on a certain nilmanifold to whose description we now turn:

2. The nilmanifold. Let $X_1, \dots, X_n$ denote the standard basis for $\mathbf{R}^n$, and define the linear map $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ by setting $TX_i = X_{i+1}$, for $1 \leq i < n$, and $TX_n = 0$. We define a semidirect product $\mathbf{R}^n \cdot \mathbf{R}$ by setting

$$(2.1) \quad (y, t)(y', t') = (y + \exp(tT)y', t + t')$$

for all $y, y' \in \mathbf{R}^n$ and all $t, t' \in \mathbf{R}$. (By $\exp(tT)$ we mean $\sum_{j=0}^{n-1} (j!)^{-1} t^j T^j$.) The group operation for $\mathbf{R}^n \cdot \mathbf{R}$ described in (2.1) is metabelian nilpotent. (This group also appears in [2, Chapter VIII].)

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² In what follows, $[\cdot]$ means "integer part" and $\{\cdot\}$ means fractional part.

Let T denote the circle group, and let $\chi: \mathbf{R}^n \rightarrow T$ be a character of $\mathbf{R}^n$. The character χ induces a unitary representation of $\mathbf{R}^n \cdot \mathbf{R}$ on $L^2(\mathbf{R})$, as follows:

$$(U_{(y,t)}^\chi f)(u) = \chi(\exp(uT)y)f(u+t)$$

for all $(y, t) \in \mathbf{R}^n \cdot \mathbf{R}$ and all $f \in L^2(\mathbf{R})$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbf{R}^n$; then there is a unique element $m \in \mathbf{R}^n$ such that $\chi(y) = \exp 2\pi i \langle m, y \rangle$. We will usually write U^m in place of U^χ .

The subset $\{(m, n!r): m \in \mathbf{Z}^n, r \in \mathbf{Z}\}$ of $\mathbf{R}^n \cdot \mathbf{R}$ is a subgroup of $\mathbf{R}^n \cdot \mathbf{R}$, denoted Γ . The quotient $M = (\mathbf{R}^n \cdot \mathbf{R})/\Gamma$ is a compact nilmanifold. (Note: We use cosets of the form Γx .) There is a unique probability measure ν on M that is invariant under translation by elements of $\mathbf{R}^n \cdot \mathbf{R}$. For each $m \in \mathbf{Z}^n$, we define (cf. [3, §3]) the "homogenizing" map $\Theta^m: L^2(\mathbf{R}) \rightarrow L^2(M, \nu)$ by

$$(\Theta^m f)(y, t) = \sum_{k=-\infty}^{\infty} (\exp 2\pi i \langle m, \exp(n!kT)y \rangle) f(t + n!k).$$

The properties of Θ^m are described in [3], all we shall need is the obvious fact that if $m \neq 0$, then $\int_M \Theta^m f d\nu = 0$.

Given an $(n+1)$ -tuple $Y = (\lambda_1, \dots, \lambda_n, \tau)$ of real numbers, we define a one-parameter subgroup of $\mathbf{R}^n \cdot \mathbf{R}$, $\exp(rY)$, $r \in \mathbf{R}$, by setting

$$y_j^{(r)} = \sum_{k=1}^j r^{k-1} \lambda_{j-k+1} / k! \quad \text{and} \quad \exp(rY) = (y_1^{(r)}, \dots, y_n^{(r)}, r\tau).$$

The main result of §5 of [1] says that the sequence $\Gamma \exp(Y), \Gamma \exp(2Y), \dots$ is uniformly distributed in M , with respect to the measure ν , if, and only if, the real numbers τ and λ_1 are linearly independent over the rational numbers.

3. Weyl's theorem. We are now ready to prove the theorem of Weyl stated in the introduction. It is a routine computation to prove that in order that Weyl's theorem be true, it is sufficient that whenever P is a polynomial over $\mathbf{R}$ whose leading coefficient is irrational, the sequence $P(1), P(2), \dots$ is uniformly distributed mod 1. Now Weyl's uniform distribution criterion (see [5]) says that for a sequence a_r of real numbers to be uniformly distributed mod 1, it is necessary and sufficient that for every nonzero integer m , we have $\lim_{N \rightarrow \infty} N^{-1} \sum_{j=1}^N \exp(2\pi i a_j m) = 0$. Thus, the proof of Weyl's theorem comes down to showing that if P is polynomial over $\mathbf{R}$ whose leading coefficient is irrational, then

$$\lim_{N \rightarrow \infty} N^{-1} \sum_{j=1}^N \exp(2\pi i m P(j)) = 0$$

for every nonzero $m \in \mathbf{Z}$. Now we appeal to nilmanifold theory.

Consider the one-parameter subgroup $\exp(rY)$, defined in §2, in the special case $\tau=n!$, $\lambda_1, \dots, \lambda_n$ arbitrary. Set $m=(0, 0, \dots, 0, \mu) \in \mathbb{Z}^n$, and let f be any C_0^∞ function on $\mathbb{R}$ with support in $[-\frac{1}{2}, \frac{1}{2}]$. Then, for any integer r ,

$$\Theta^m f(\exp(rY)) = f(0) \exp 2\pi i \langle m, \exp(-n!rT)y^{(r)} \rangle.$$

Now

$$\begin{aligned} \langle m, \exp(-n!rT)y^{(r)} \rangle &= \mu \sum_{j=0}^{n-1} (-1)^j (n!r)^j y_{n-j}^{(r)} / j! \\ &= \mu \sum_{j=0}^{n-1} (-1)^j (j!)^{-1} (n!r)^j \sum_{i=1}^{n-j} (n!r)^i (n!i!)^{-1} \lambda_{n-j-i+1} \\ &= \mu \sum_{k=1}^n \lambda_{n-k+1} (n!r)^k (n!k!)^{-1} \sum_{j=0}^{k-1} (-1)^j k! (j!(k-j)!)^{-1} \\ &= \mu \sum_{k=1}^n (-1)^{k+1} (n!)^{k-1} (k!)^{-1} \lambda_{n-k+1} r^k. \end{aligned}$$

Set $a_k = (-1)^{k+1} (n!)^{k-1} (k!)^{-1} \lambda_{n-k+1}$, and define the polynomial Q over $\mathbb{R}$ by setting $Q(X) = a_1 X + \dots + a_n X^n$. Then what we have shown is that $\Theta^m f(\exp(rY)) = f(0) \exp(2\pi i \mu Q(r))$. Assume that λ_1 (or equivalently, a_n) is irrational; then the sequence $\exp(Y)$, $\exp(2Y)$, $\dots$ is uniformly distributed in M , and therefore

$$\lim_{N \rightarrow \infty} N^{-1} f(0) \sum_{r=1}^N \exp(2\pi i \mu Q(r)) = \int_M \Theta^m f \, d\nu = 0.$$

It follows that the sequence $Q(1)$, $Q(2)$, $\dots$ is uniformly distributed mod 1. Since $\lambda_1, \dots, \lambda_n$ were arbitrary, except for the condition that λ_1 be irrational, our proof of Weyl's theorem is complete.

4. Powers of fractional parts. We next consider what happens when we choose for $\exp(rY)$ the subgroup got when τ is irrational, $\lambda_1=1$, and $\lambda_2=\lambda_3=\dots=\lambda_n=0$. Taking $m=(0, 0, \dots, 0, \mu) \in \mathbb{Z}^n$ and $f \in C_0^\infty[0, 1]$, we get

$$\Theta^m f(\exp(rY)) = f(\{\tau r\}) \exp 2\pi i \langle m, \exp(-[\tau r]T)y^{(r)} \rangle,$$

where $\{\cdot\}$ = the fractional part, and $[\cdot]$ = the integer part. Now

$$\begin{aligned} \langle m, \exp(-[\tau r]T)y^{(r)} \rangle &= \mu \sum_{j=0}^{n-1} (-1)^j (j!)^{-1} [\tau r]^j y_{n-j}^{(r)} \\ &= \mu (\tau n!)^{-1} \sum_{j=0}^{n-1} n! (j!(n-j)!)^{-1} [\tau r]^j (\tau r)^j \\ &= \mu (\tau n!)^{-1} (\{\tau r\}^n + (-1)^{n+1} [\tau r]^n). \end{aligned}$$

Because $\exp(rY)$, $r=1, 2, \dots$, is uniformly distributed in M , and because $\int_M \Theta^m f d\nu = 0$, we have that for every $\varepsilon > 0$ there is an $N_{\varepsilon, f} > 0$ such that whenever $N > N_{\varepsilon, f}$,

$$(4.1) \quad \left| N^{-1} \sum_{r=1}^N \Theta^m f(\exp(rY)) \right| < \varepsilon.$$

Choose $f \in C_0^\infty[0, 1]$ so that $0 \leq f \leq 1$ and so that $f(x) = 1$ for all $x \in [\varepsilon/2, 1 - \varepsilon/2]$, and set

$$S(\varepsilon, N) = \{r \in \mathbb{Z} : 0 < r \leq N \text{ and } \{\tau r\} \in [0, \varepsilon/2] \cup [1 - \varepsilon/2, 1]\}.$$

Because the sequence $\tau, 2\tau, 3\tau, \dots$ is uniformly distributed mod 1, there is an $N'_\varepsilon > 0$ such that whenever $N > N'_\varepsilon$, the number of elements in $S(\varepsilon, N)$ is at most $2\varepsilon N$.

Consider now the average

$$A_N = N^{-1} \sum_{r=1}^N \exp(2\pi i \mu (\tau n!)^{-1} (\{\tau r\}^n + (-1)^{n+1} [\tau r]^n)).$$

We wish to prove that if $\mu \neq 0$, then $\lim_{N \rightarrow \infty} A_N = 0$. Set

$$A'_N = N^{-1} \sum_{r \in S(\varepsilon, N)} \exp(2\pi i \mu (\tau n!)^{-1} (\{\tau r\}^n + (-1)^{n+1} [\tau r]^n)),$$

and set $A''_N = A_N - A'_N$. Because for N large enough, $S(\varepsilon, N)$ contains at most $2\varepsilon N$ elements, we have $|A'_N| < 2\varepsilon$ for N large. As for A''_N , our choice f was made so that $A''_N = N^{-1} \sum_{r \notin S(\varepsilon, N)} \Theta^m f(\exp(rY))$. It follows, therefore, from the inequality (4.1) that for large N we have $|A''_N| < 3\varepsilon$. Hence $|A_N| < 5\varepsilon$ for large N , and we have $\lim_{N \rightarrow \infty} A_N = 0$, as desired. We have thus proved:

THEOREM. *Let τ be an irrational number, and let n be an integer greater than 1; then the sequence*

$$a_r = (\tau n!)^{-1} (\{\tau r\}^n + (-1)^{n+1} [\tau r]^n)$$

is uniformly distributed mod 1.

We remark that the case $n=2$ reduces to the uniform distribution mod 1 of τr^2 , which is not surprising in view of the structure of $\mathbb{R}^2 \cdot \mathbb{R}$, as a comparison between §3 and this section will show. The first interesting case is $n=3$, which again is not surprising, as Hahn's "torus" arguments cease to apply in that case.

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