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A FACTORIZATION THEOREM FOR COMPACT OPERATORS

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ABSTRACT. It is shown that every compact operator $T:E\to F$ between Banach spaces admits a compact factorization ($T=QP$ where $P:E\to c$ and $Q:c\to F$ are compact) through a closed subspace c of the Banach space c_0 of zero-convergent sequences.

A (linear) operator $T:E\to F$ between Banach spaces is *compact* if T transforms the unit ball of E into a relatively compact subset of F. The author has recently shown [**3**, Corollary 2.10] that an operator $T:E\to F$ is compact if and only if there is a sequence λ in c_0 and a sequence $\{a_n\}$ in the unit ball of the topological dual E' of E such that

$$\|Tx\| \leqq \sup |\lambda_n|\, |\langle x, a_n\rangle|$$

for all x in E.

THEOREM. *If $T:E\to F$ is a compact operator between Banach spaces, then there is a closed subspace c of c_0 and compact operators $P:E\to c$ and $Q:c\to F$ with $T=QP$.*

PROOF. Suppose that $T:E\to F$ is compact. Then there is a sequence λ in c_0 and a sequence $\{a_n\}$ in E' such that for each x in E

$$(*) \qquad \|Tx\| \leqq \sup |\lambda_n|^2\, |\langle x, a_n\rangle|.$$

Let $P:E\to c_0$ be the compact operator defined by $P(x)=\{\lambda_n\langle x, a_n\rangle\}$. Let c denote the closure of $P(E)$ in c_0. Let $D:c\to c_0$ be the compact operator defined by $D(\xi)=\{\lambda_n\xi_n\}$. Let $S:D(c)\to F$ be the (unique) bounded (by $(*)$, $\|S\|\leqq 1$) operator such that $S(DPx)=T(x)$ for all x in E. Let $Q=SD$. Then $T=QP$, where both Q and P are compact.

REMARK 1. Grothendieck [**1**, Chapitre I, p. 165] has shown that a Banach space E has the approximation property if, for every Banach space F and every compact operator $T:F\to E$, there exists a sequence of finite rank operators $T_n:F\to E$ with $\|T_n-T\|\to 0$. This result together with our factorization theorem can be used to give an elementary proof

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of the following result of Grothendieck [**1**, Chapitre I, pp. 170–171]: If each closed subspace c of c_0 has the approximation property, then every Banach space has the approximation property.

REMARK 2. Lindenstrauss and Tzafriri [**2**, p. 265] have recently shown that a Banach space E is isomorphic to a Hilbert space if and only if, for every closed subspace F of E, every compact operator $T:F\to F$ can be extended to a bounded operator $S:E\to F$. By combining this result with our factorization theorem it follows that a Banach space E is isomorphic to a Hilbert space if and only if, for every closed subspace F of E and every closed subspace c of c_0, every compact operator $T:F\to c$ can be extended to a bounded operator $S:E\to c$. This result contrasts with the following (unpublished) result of the author: If E is an infinite dimensional Hilbert space, then there exists a closed subspace c of c_0 and a compact operator $T:c\to E$ that cannot be extended to a bounded operator from c_0 into E.

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