PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 34, Number 1, July 1972

LOCAL DEGREE OF SEPARABILITY AND INVARIANCE OF DOMAIN

L. B. TREYBIG

ABSTRACT. In E^n an invariance of domain theorem may be proved assuming the Jordan Brouwer Theorem. In this paper such a theorem is proved for various locally compact, connected, Hausdorff spaces which satisfy a certain local degree of separability property. An example shows the separability condition may not be removed. A second theorem yields additional information about homogeneous spaces which satisfy the hypotheses of the first theorem.

In ([2], [3], [4]) the invariance of domain for n-manifolds is proved using either essential mappings or the Jordan Brouwer Theorem. The latter proof is generalized in Theorem 1 to certain locally compact, connected Hausdorff spaces by adding hypotheses concerning local degree of separability. Curiously enough, such a condition is necessary in the sense that there is a counterexample (Example 1) to Theorem 1 if the separability condition is omitted. Theorem 2 shows that if a homogeneous space X satisfies the conditions of Theorem 1 plus two other restrictions, then X is first countable and locally separable.

The space X will be said to have the *invariance of domain property* if given $h: U\to X$ a homeomorphism of an open subset U of X into X, then $h(U)$ is open. The *local degree of separability*, l.s.(p), of X at $p \in X$ is the least cardinal k such that an open neighborhood of p contains a dense subset B with card $B \leqq k$.

THEOREM 1. *Let (X, T) be a locally compact, connected Hausdorff space such that if $a \in U \in T$ and $b \in X-U$, then there is a collection C of mutually exclusive continua such that* (1) *$a \in \bigcup C \subset U$, where $\bigcup C$ is connected and open*, (2) *if $a \in g_0 \in C$ and $g \in C-\{g_0\}$, then g separates a from b in X*, (3) *if $h: \bigcup C \to X$ is a homeomorphism into and $g \in C-\{g_0\}$ then g contains a subcontinuum g' such that $X-h(g')$ is not connected, and* (4) card $C >$ l.s.(p) *for each $p \in X$. Then X has the invariance of domain property.*

Received by the editors March 5, 1970.

AMS 1969 *subject classifications*. Primary 5460; Secondary 5440, 5478.

Key words and phrases. Invariance of domain, local degree of separability, homogeneous space.

PROOF. Suppose $U \in T$ and $h: U \to X$ is a homeomorphism into, but that $y \in h(U) \cap \mathrm{Cl}(X-h(U))$. We may also suppose without loss of generality that U is connected. Let $x = h^{-1}(y)$ and $W \in T$ such that $y \in W \subset \overline{W} \subset X-z$, where $z \in h(U)$, and $\overline{W}$ and $\overline{W} \cap h(U)$ are both compact.

By the hypothesis there is a connected open set W_1 so that $y \in W_1 \subset \overline{W}_1 \subset W$. Some subcontinuum A of $\overline{W}_1$ is irreducible between a point t of $\overline{W}_1 - h(U)$ and $\overline{W}_1 \cap h(U)$. $A - h(U)$ is connected and has a point s of $h(U)$ in its closure. Letting $s=a$, $b=t$, $U-t$, we find a collection C' of continua as guaranteed in the hypothesis. But since $\bigcup C'$ is open, some element B of C' separates s from t in X and also intersects $A-h(U)$ and $h(U) \cap \overline{W}$. Some subcontinuum B' of B is irreducible between a point of $B \cap (A-h(U))$ and $B \cap h(U)$. $B'-h(U)$ has a point r of $h(U)$ in its closure, where $r \neq s$. Thus $D = (A \cup B') - h(U)$ is a connected subset of $X-h(U)$ with points r, s of $h(U) \cap W$ in its closure.

Let $s \in M \in T$ where M has a dense subset N where card $N = \mathrm{l.s.}(s)$. Let a, b, $V = \bigcup C$, and C be as in the hypothesis where $a = h^{-1}(s)$, $b = h^{-1}(r)$, and $V \subset U \cap (h^{-1}(M \cap W_1 - r))$. Assume $a \in g_0 \in C$, and for each $g \in C - \{g_0\}$ let g' denote a subcontinuum of g such that $X - h(g')$ is the union of two mutually separated sets $R_{g'}$ and $S_{g'}$, where $s \notin R_{g'}$.

Now suppose g_1 and g_2 are two elements of C such that g_1 separates g_2 from a in V. (Note that the methods of Theorem 81, p. 33 of [5] reveal that $C - \{g_0\}$ is totally ordered under the relation $g < g'$ if and only if g separates a from g' in X; in fact, with the topology induced by $< C - \{g_0\}$ it is also connected.) If $R_{g_1'}$ and $R_{g_2'}$ intersect, then $h(g_1') \not\subset R_{g_2'}$; for otherwise it would then follow that g_2' separates g_1' from a in U; and thus g_2 would separate g_1 from a in X, a contradiction. Therefore $h(g_2') \cup R_{g_2'} \subset R_{g_1'}$. Let U_1 denote the complementary domain of $U - g_2$ containing $h^{-1}(r)$. But $D \cup \{r, s\} \cup h(U_1 \cup g_2)$ is a connected subset of $X - h(g_1')$ which contains s and a point of $R_{g_1'}$, a contradiction. Thus $R_{g_1'} \subset X - R_{g_2'}$.

Finally, $L = \{M \cap R_{g'} : g \in C\}$ is a collection of disjoint open subsets of M where card $L = \text{card } C > \mathrm{l.s.}(s)$. Since each element of L contains an element of N it follows that card $N \geqq$ card C, so $\mathrm{l.s.}(s) \geqq$ card C, a contradiction.

COROLLARY 1. *A locally compact Moore space satisfying Axioms* 0–5 *of* [5] *has the invariance of domain property.*

PROOF. This follows from Theorem 1 with the aid of Theorem 58, p. 23 and Theorem 14, p. 171 of [5].

REMARK. In Theorem 1 if X is a n-manifold, then for each $x \in U$ open let k be a homeomorphism from $V \subset U$ onto the open unit ball in R^n, where $k(x)=0$. Let $C = \{\{x\}\} \cup \{k^{-1}(S)$: S is a sphere in R^n with center 0 and radius less than 1$\}$.

LEMMA 1. *Suppose a, b, U, C, $V=\bigcup C$ are as in the hypothesis of Theorem* 1, *and $a \in g_0 \in C$ and $C-\{g_0\}$ is totally ordered under the relation $\leqq$ described above. Then, if for each $g \in C-\{g_0\}$, the set $X-g=R_g \cup S_g$ mutually separated where R_g is the component of $X-g$ containing a then* (1) *there is a $g \in C-\{g_0\}$ so that if $g' \leqq g$ then $g' \cup R_{g'} \subset V$ and* (2) *if W is an open set containing g_0 then there exists $g \in C-\{g_0\}$ such that $R_g \cup g \subset W$.*

PROOF. Since X is locally connected, there is no harm in assuming each R_g above is the component of $X-g$ containing a. Note from above that $g<g'$ implies $g \cup R_g \subset R_{g'}$. Let $M=\bigcap_{g \in C-\{g_0\}} R_g=\bigcap_{g \in C-\{g_0\}} g \cup R_g=\bar{M}$. Suppose $M-g_0$ is not void. Since $M-g_0 \subset X-V$ and g_0 is closed, then g_0 and $M-g_0$ are mutually separated. Since X is connected let $x \in (M-g_0) \cap \mathrm{Cl}(\bigcup S_g)$. Let W_0 be a connected open set containing x so that $\bar{W}_0$ is a compact subset of $X-g_0$ and let $g \in C-\{g_0\}$ such that W_0 intersects S_g. But W_0 must intersect g since otherwise $W_0 \subset S_g$. Thus W_0 intersects $S_{g'}$ for all $g' \leqq g$. Using connected open subsets of $X-g_0$ whose closures are compact, and which intersect $g \cup S_g$, a chain argument yields a continuum N so that $b, x \in N \subset X-g_0$. Thus if $g' \leqq g$ then g' intersects N.

There is an open set R containing g_0 so that $\bar{R}$ is a compact subset of $X-(N \cup M-g_0)$. For every $g_1 \leqq g$ there is a $g' \leqq g_1$ so that g' intersects R and also N. Thus, there is a point t of $\mathrm{Bd}\, R$ so that if $t \in Q \in T$ and $g_1 \leqq g$ then there exists $g' \leqq g_1$ such that g' intersects Q. Since $t \notin M$, $t \in S_{g'}$ for some g'. But if $g''<g'$ then g'' does not intersect $Q=S_{g'}$, a contradiction. Thus $g_0=M$.

In part (2) suppose W_1 is an open set such that $g_0 \subset W_1 \subset W$, where $\bar{W}_1$ is compact. There is a finite set $\{g_1, \cdots, g_n\}$ of elements of $C-\{g_0\}$ such that $\bigcup_{p=1}^{n} S_{g_p}$ covers $\mathrm{Bd}(W_1)$. Let g_i' denote the least of these in the order $\leqq$. Since $g_i \cup R_{g_i}$ is connected and contains a but no point of $\mathrm{Bd}(W_1)$, then $g_i \cup R_{g_i} \subset W_1 \subset W$.

THEOREM 2. *If* (1) (X, T) *is as in Theorem* 1 *and is homogeneous,* (2) $\aleph_1=2^{\aleph_0}$ *and* (3) *for each such a, b, U described in Theorem* 1 *the element g_0 of C which contains a is $\{a\}$, then X is locally separable and first countable.*

PROOF. Let $a, b, U, C, V=\bigcup C$ be as in Theorem 1 where $\bar{V}$ is compact and $a \in g_0 \in C$. Let $g, g_1, g_2, \cdots$ be a sequence of elements of $C-\{g_0\}$ such that $g<g_{p+1}<g_p$, $p=1, 2, \cdots$. There exists $g' \in C-\{g_0\}$ so that g' is the g.l.b.$\{g_1, g_2, \cdots\}$ and a point x of g' so that every open set containing x intersects infinitely many g_i's.

Let C' be as in Theorem 1 for $a'=x$, $b'=b$, $U'=U$, and let $V'=\bigcup C'$ and $x \in h_0 \in C'$. For each n let $U_n=X-g_n$ and for each $h \in C'-\{h_0\}$ let $X-h=R_h \cup S_h$ mutually separated, where $x \in R_h$ and R_h is connected. Let $C'-\{h_0\}$ be ordered as above. Let elements $h_1, h_2, \cdots$ of $C'-\{h_0\}$ be chosen such that $h_n \cup R_{h_n} \subset U_n$ and $h_{n+1}<h_n$ for $n=1, 2, \cdots$.

Suppose $x \in Q \in T$. But by Lemma 1 (since $\{x\}=h_0$) there is an n so that $R_{h_n} \cup h_n \subset Q$. Thus, X has a countable base at x, so by homogeneity has one at each point.

Let $y \in g'$ such that every open set containing y intersects a g'' for $g''<g'$, and let $R_1, R_2, \cdots$ denote a countable base at y. Select elements $k_1, k_2, \cdots$ of $C-\{g_0\}$ such that k_n intersects R_n and $k_n<k_{n+1}$. The open segments (k_n, g_n) form a countable base at g' in the connected totally ordered set $C-\{g_0\}$. Analogous double use of the countable base at a point in X will produce for any $g'' \in (C-\{g_0\}, \leqq)$ a countable base. By a theorem of Babcock [**1**], card$(C-\{g_0\}) \leqq 2^{\aleph_0}$. By the continuum hypothesis l.s.$(p) \leqq \aleph_0$ for each $p \in X$.

Background. Given a well-ordered sequence α and a totally ordered set B let B^α denote the set of all sequences isomorphic to α, each term of which is in B, and let B^α be understood to have the lexicographic order. Let $L_0=L=[0, 1]$ and let $\alpha_1=1, 2, 3, \cdots$. Also, let $\alpha_2={\alpha_1}^{\alpha_1}$ and let $L_i=L^{\alpha_i}$ $(i=1, 2)$.

It is known (Babcock [**1**]) that if J denotes one of L_0, L_1, and L_2, then in the interval topology (1) J is compact, connected, and first countable, and (2) every pair of subintervals of J are homeomorphic. Furthermore, no two of L_0, L_1 and L_2 are homeomorphic. Let $L_p=a_pb_p$, $p=0, 1, 2$.

Lemma 2. *Let G denote an upper semicontinuous decomposition of $L_2 \times L_2$ such that $g \in G$ provided* (1) *$g=\{(a, b)\}$ where $a, b \in L_2$ and $b>a_2$, or* (2) *there is an element (a, a_2) of $L_2 \times L_2$ such that $g=\{(a', a_2)$ in $L_2 \times L_2$ so that a and a' agree on all coordinates not preceded by an infinite number of coordinates$\}$. Then, there is no homeomorphism of $(L_2 \times L_2)/G$ into $L_2 \times L_2$.*

Proof. Suppose there is such a homeomorphism h. Let $c_1, c_2, \cdots$ denote a sequence of elements of L_2 which converge to a_2, and where $c_{p+1}<c_p$ for $p=1, 2, \cdots$. Let $d_p=h(L_2 \times \{c_p\})$, $p=1, 2, \cdots$ and let $d_0=$ image of the nondegenerate elements of G under h. Since d_0 is homeomorphic to L_1, d_0 contains no interval of the form $\{a\} \times K$ or $H \times \{b\}$, so let e denote a "subarc" of d_0 containing no points with a coordinate $=a_2$ or b_2.

For each n let G_n denote a finite cover of e by sets of the form $P=H \times K$, where each of H and K is an open subinterval of L_2, and where $P \subset L_2 \times L_2-d_n$. Let C_n denote the set of all components C of sets of the type $e \cap P$, $P \in G_n$, and let K_C denote a set composed of the endpoints of C and one point interior to C. For each n, let $H_n=\bigcup K_C$, $C \in C_n$.

In order to show each C_n is countable it is helpful to use (1) the fact that $L_2 \times L_2$ is first countable and (2) the fact that no generalized arc A has the property that there are mutually exclusive closed sets M, N and an infinite set T of mutually exclusive segments of A such that each $t \in T$ has one endpoint in M and the other in N. Finally, to show $\bigcup H_n$ is dense in e it must be remembered that e contains no "vertical" or "horizontal" intervals. Since $\bigcup H_n$ is a countable set dense in e, this means L_1 is homeomorphic to L_0, a contradiction.

EXAMPLE 1. There is a space (X, T) satisfying all but condition (4) of the hypothesis of Theorem 1, and such that X does not have the invariance of domain property.

PROOF. Before we describe the example we need to describe some further decompositions of $L_2 \times L_2$. Let G be as in Lemma 2. Let H be a decomposition of $L_2 \times L_2$ so that H agrees with G on points (a, b) with $b < b_2$, but on $L_2 \times \{b_2\}$ let (a, b_2) and (a', b_2) belong to the same element of H if and only if a and a' have the same first coordinate. Let K be defined so that $g \in K$ if and only if (1) g is an element of H containing no point of the form (a_2, x) or (b_2, x), or (2) there is an x in L_2 so that g is the union of the elements of H containing (a_2, x) and (b_2, x), respectively. The set $A = (L_2 \times L_2)/K$ is a "generalized annulus" with a metric simple closed curve on one edge E_J^0 and a "simple closed curve" on the other edge E_J^1, which is the union of two I_1 arcs. Given a subset M of A let $P_1(M)$ denote the set of all elements k of K so that there is an element m of M, where k contains an element of the form (a, a_2) and m contains an element of the form (a, x). Likewise, define $P_0(M)$ for points on the other edge. Note that if m is a subset of the metric edge, and N is the set of all points (x, a_2) so that $(x, b_2) \in m \in M$, then N is the union of elements of K.

The space X will denote the Euclidean plane R^2 together with the union of a set of "annuli" A_J, one for each simple closed curve J in the plane. The metric edge of A_J is identified with J under an identification map i_J: $E_J^0 \to J$, and if $J \neq J'$, then $A_J \cap A_{J'} = J \cap J'$.

The topology T for X is generated by neighborhoods of the following type: If $x \in A_J - J$, let small open neighborhoods of x be those in the decomposition space topology on A_J. If $x \in R^2$, a neighborhood U of x will be determined by (1) an $\varepsilon > 0$, (2) the collection V of all simple closed curves J which intersect the spherical open set $N(x, \varepsilon)$, and (3) a collection W of connected open subsets S_J, one for each L_{2J} (Jth copy of L_2) such that $J \in V$ and such that (1) S_J contains the b_{2J} endpoint and (2) $S_J = L_{2J}$ for all but finitely many J's in W. U is $\{p : (1)\, p \in N(x, \varepsilon)$ or (2) there is a $J \in V$, a point $q \in J \cap N(x, \varepsilon)$, and a point (r, s) of $I_{2J} \times I_{2J}$ such that $(r, s) \in p$, $s \in S_J$, and $i_J(P_0(p)) = q\}$.

We now see how to define the various collections C of continua. Let $a \in U$ open and $b \in X - U$.

Case 1. Suppose $a \in E_J^1$. We think of L_2 as the Jth copy and of K as the corresponding decomposition of $L_2 \times L_2$. Let x_1, x_2 be two elements of E_J^1 distinct from a, suppose $a_2 < W < b_2$ and suppose B is the "arc" from x_1 to x_2 on E_J^1 that contains a. Let g be the $\{P : (1)\, P = \{(x, w)\}$ and $P_1(P) \in B$, or (2) $P=\{(x, y)\}$ and $a_2<y\leqq W$ and $P_1(P)=x_1$ or x_2, or (3) $P=x_1$ or $x_2\}$. Continua such as g (type g) will be used to construct C, although not all continua in C will be of this type.

Let U_1, $U_2, \cdots$ denote a countable base of neighborhoods at a, where $U_1\subset U$. Let g_0 be a continuum of type g so that $g\cup$ (the component of A_J-g that contains $a)\subset U_1$. Let $g_1=\{a\}$ and let $g_{1/2}$ be a type g continuum so that $g_{1/2}\subset U_2$ and also separates g_0 from a in A_J. Analogously, we find $g_{1/4}$ and $g_{3/4}$ so that $g_{3/4}\subset U_3$ and separates $g_{1/2}$ from a and where $g_{1/4}$ separates g_0 from $g_{1/2}$. This process is continued to find for each $r=p/2^q$ $(0\leqq r<1)$ a continuum of type g, where the separations occur in the same way as on the real line, and where $g_r\subset U_{q+1}$ for $r=2^q-1/2^q$. If $0<t<1$ and $t\neq p/2^q$ then g_t is the set of all points of A_J that are separated from g_0 by a previously defined g_s, for $s<t$, but are not separated from g_0 by such a g_s for $s>t$. $C=\{g_t:1\geqq t>0\}$.

Case 2. If $a\in A_J-(E_J^0\cup E_J^1)$, then a proof analogous to that in Case 1 may be used. The continua will have four "sides" instead of three.

Case 3. Suppose $a\in R^2$ and let $U_1\subset U$ be determined by ε, V, and W as in the definition of this type of neighborhood above. Let $s_{J_1}, \cdots, s_{J_n}$ be the sets in W which are different from the corresponding L_J. For each J_p $(p=1, \cdots, n)$ let h_p denote a set valued map so that if $t\in[0, \varepsilon]$ then $h_p(t)$ is the set of all w in I_{2J_p} whose first coordinate is $(1/\varepsilon)(\varepsilon c_p+(d_p-c_p)t)$ $(c_p<d_p)$ and where every point in L_{2J_p} with first coordinate in $[c_p, d_p]$ is in s_{J_p}.

For $1>t>0$ let $g_t=\{P:$(i) $P\in R^2$ and $|P-a|=\varepsilon t$, or (ii) $P\in A_J$, $J\in W$, $s_J=I_{2J}$, and $|P_J^0(P)-a|=\varepsilon t$, or (iii) $P\in J_m$ $(1\leqq m\leqq n)$ and (a) $|P_{J_m}^0(P)-a|=\varepsilon t$ and $P=\{(x, y)\}$, where $y\geqq V\in h_m(t\varepsilon)$, or (b) there is a component C of $J_m-\{Q\in R^2:|Q-a|=\varepsilon t\}$ such that $P=\{(x, y)\}$ and $P_{J_m}^0(P)\in C$ and $y\in h_m(t\varepsilon)\}$. The set g_0 is defined to be the closure of the component of $X-\bigcup g_t$ $(0<t<1)$ that contains a.

To verify that condition (3) of the hypothesis holds, note that an application of Lemma 2 reveals that if $h: V\rightarrow X$ is a homeomorphism into, where U is an open subset of a A_J-J containing a segment s of E_J^1, then $h(s)$ is a segment of some $E_{J'}^1$. That $h(g_t)$ (Case 1) separates X is a consequence of the work of Slye [6] applied to two sets of the form $A_J-E_J^0$ joined along a common edge E_J^1. In Case 2 the work of Slye may be applied to $A_J-(E_J^0\cup E_J^1)$. In Case 3 each g_t $(0<t<1)$ contains a simple closed curve $J\subset R^2$, and $h(J)$ separates X into A_J-J and $X-A_J$.

Let $W=(-1, 1)\times(-1, 1)\subset R^2$ be an open square disk and let Q denote the set of all closed curves in R^2 which intersect W. Let $U=\{P\in X:(1)$ $P\in W$ or (2) $P\in A_J$, $J\in Q$ and $P_J^0(P)\in W\}$. For each positive integer n let J_n be the rectangular simple closed curve with vertices at $(-n, 0)$, $(n, 0)$, (n, n) and $(-n, n)$, respectively, and let C_n denote the set of all points P

of A_{J_n} so that $P^0_{J_n}(P) \in W$. We define an into homeomorphism $h: U \to U$ which is the identity on $U - \bigcup C_n$ and such that $h(C_n) = C_{n+1}$. But $U - (C_1 - J_1)$ is not open, so X does not have the invariance of domain property.

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Department of Mathematics, Tulane University, New Orleans, Louisiana 70118

Current address: Department of Mathematics, Texas A & M University, College Station, Texas 77843