ARITHMETIC STRUCTURE AND LACUNARY FOURIER SERIES

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ABSTRACT. We prove a theorem concerning the arithmetic structure of $\Lambda(p)$ sets. This generalizes a result of Rudin and yields a new characterization of Sidon sets for certain Abelian groups.

Let G be a compact Abelian group with dual group Γ . A subset $E \subset \Gamma$ is called a $\Lambda(p)$ set ($2 < p < \infty$) if for every trigonometric polynomial $f = \sum_1^n a_k \gamma_k$ ($\gamma_1, \dots, \gamma_n \in E$) an inequality $\|f\|_p \leq B \|f\|_2$ holds, where the constant B depends only on E and p . Theorems 1 and 2 of this paper give estimates on the number of elements of E which can lie in a given multi-dimensional arithmetic progression in Γ . Theorem 2 is a generalization of Rudin's result [5, Theorem 3.5]. When a set E is $\Lambda(p)$ for all $2 < p < \infty$ and the constant B may be taken as $Cp^{1/2}$ where C depends only on E then it is called a Λ set. A subset $E \subset \Gamma$ is said to be a Sidon set if every bounded function on E can be extended to be a Fourier-Stieltjes transform. It is known that every Sidon set is a Λ set, but the converse is unknown [4, p. 128]. As a corollary to Theorem 2 we find that Λ sets are also Sidon sets in groups Γ where each element has the same prime order.

NOTATION. Suppose s and N are positive integers, $1 \leq r < \infty$, $b_1, \dots, b_k \in \Gamma$ have finite order, $b_{k+1}, \dots, b_s \in \Gamma$ have infinite order and $b_0 \in \Gamma$ has arbitrary order. Set

$$A_r(N, s, b) = \left\{ n \cdot b + b_0 \left| \sum_{k=1}^s |n_j|^r \leq N^r \right. \right\}$$

where $n \cdot b = n_1 b_1 + \dots + n_s b_s$. Denote by $\hat{f}$ the Fourier transform of $f \in L^1(G)$, by $\langle x, \gamma \rangle$ the action of the character γ on $x \in G$, and by $|F|$ the cardinality of a finite set F .

THEOREM 1. Let $b_1, \dots, b_k \in \Gamma$ have finite orders $\beta_1, \dots, \beta_k$, and let $b_{k+1}, \dots, b_s \in \Gamma$ have infinite order. Let $E \subset \Gamma$ be a $\Lambda(p)$ set with B a

Received by the editors September 24, 1971 and, in revised form, December 8, 1971.
AMS 1970 subject classifications. Primary 42A44, 43A70.

Key words and phrases. Lacunary Fourier series, Sidon sets, $\Lambda(p)$ sets.

¹ This work constitutes part of the author's doctoral dissertation at the University of Maryland under the direction of Professor J. J. Benedetto.

constant as in the definition of $\Lambda(p)$ set. If $A_2(N, s, \mathbf{b}) = A_2$ is as defined above and $b_{k+1}, \dots, b_s$ are independent in Γ then

$$|A_2 \cap E| \leq B^2 e^2 (\beta_1 \cdots \beta_k)^{2/p} (1 + N(\pi/2)^{1/2})^{2(s-k)/p}.$$

PROOF. Since translates of $\Lambda(p)$ sets are $\Lambda(p)$ sets with the same constant of definition, we may take $b_0 = 0$. Define

$$\begin{aligned} g_{b_j}(x) &= \sum_{n=0}^{\beta_j-1} \langle x, nb_j \rangle & \text{for } j = 1, \dots, k, \\ g_{b_j}(x) &= \sum_{n \in \mathbb{Z}} \exp(-n^2/N^2) \langle x, nb_j \rangle & \text{for } j = k+1, \dots, s. \end{aligned}$$

It is easy to check that $g_{b_j}(x) \geq 0$ for all $x \in G$.

Let $\Gamma' = \mathbb{Z}(\beta_1) \oplus \cdots \oplus \mathbb{Z}(\beta_k) \oplus \mathbb{Z}^{s-k}$, then

$$g(x) = \sum_{\mathbf{n} \in \Gamma'} \exp(-\|\mathbf{n}^*/N\|_2^2) \langle x, \mathbf{n} \cdot \mathbf{b} \rangle$$

where $\mathbf{n}^* = (0, \dots, 0, n_{k+1}, \dots, n_s)$. The map $\phi(\mathbf{n}) = \mathbf{n} \cdot \mathbf{b}$ is a homomorphism of Γ' into Γ . If $H_\gamma = \phi^{-1}(\gamma)$ then

$$\hat{g}(\gamma) = \sum_{\mathbf{n} \in H_\gamma} \exp(-\|\mathbf{n}^*/N\|_2^2)$$

when $\gamma \in \text{Im } \phi$ and $\hat{g}(\gamma) = 0$ otherwise.

We will show that

$$(1) \quad \hat{g}(0) \leq e \hat{g}(\gamma) \quad \text{for all } \gamma \in A_2(N, s, \mathbf{b}).$$

Define an equivalence relation on H_0 by h_1 is related to h_2 if $h_1 = \pm h_2$. Denote the quotient space by $\mathcal{H}$ and the class containing h by $[h]$. Let $\gamma \in \text{Im } \phi$ be fixed and take $\mathbf{n}_\gamma \in H_\gamma$, then since $H_\gamma = \mathbf{n}_\gamma + H_0$ we have

$$\begin{aligned} \hat{g}(\gamma) &= \sum_{\mathbf{m} \in H_0} \exp(-\|(\mathbf{m}^* + \mathbf{n}_\gamma^*)/N\|_2^2) \\ &= \sum_{[\mathbf{m}] \in \mathcal{H}} \sum_{\mathbf{m} \in [\mathbf{m}]} \exp(-\|\mathbf{m}^*/N\|_2^2 - \|\mathbf{n}_\gamma^*/N\|_2^2 - 2\mathbf{m}^* \cdot \mathbf{n}_\gamma^*/N^2) \\ &= \exp(-\|\mathbf{n}_\gamma^*/N\|_2^2) \left(1 + 2 \sum_{[\mathbf{m}] \in \mathcal{H} - \{0\}} \exp(-\|\mathbf{m}^*/N\|_2^2) \cosh(2\mathbf{m}^* \cdot \mathbf{n}_\gamma^*/N^2) \right). \end{aligned}$$

Hence

$$\begin{aligned} \hat{g}(0) &= 1 + 2 \sum_{[\mathbf{m}] \in \mathcal{H} - \{0\}} \exp(-\|\mathbf{m}^*/N\|_2^2) \\ \hat{g}(\gamma) &\geq e^{-1} \left(1 + 2 \sum_{[\mathbf{m}] \in \mathcal{H} - \{0\}} \exp(-\|\mathbf{m}^*/N\|_2^2) \right), \quad \text{for } \|\mathbf{n}_\gamma^*\|_2 \leq N. \end{aligned}$$

Therefore (1) follows.

Let f be the trigonometric polynomial such that $\hat{f}(\gamma)=1$ for all $\gamma \in E \cap A_2$ and $=0$ otherwise. Let $\alpha=|E \cap A_2|$ and $\delta=\min\{\hat{g}(\gamma)|\gamma \in A_2\}$. Then

$$\begin{aligned}\alpha &= \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \\ &\leq \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \hat{g}(\gamma) / \delta = (1/\delta) \int_G f(x) g(x) dx \\ &\leq (1/\delta) \|f\| \|g\|_{qp}\end{aligned}$$

where $1/p+1/q=1$. By the logarithmic convexity of the $\| \cdot \|_q$ norms and the fact that $\|g\|_1=\hat{g}(0)$, we have

$$\|g\|_q \leq \|g\|_1^{(2-q)/q} \|g\|_2^{(2q-2)/q} = \hat{g}(0)(\|g\|_2/\|g\|_1)^{2/p}.$$

Therefore $\alpha \leq \|f\|_p (\hat{g}(0)/\delta)(\|g\|_2/\|g\|_1)^{2/p}$. It follows from the definition of $\Lambda(p)$ set and the estimate (1) that $\alpha \leq B\|f\|_2 e(\|g\|_2/\|g\|_1)^{2/p}$ and since $\|f\|_2=\|\hat{f}\|_2=\alpha^{1/2}$,

$$\alpha \leq B^2 e^2 (\|g\|_2/\|g\|_1)^{4/p}.$$

It remains to estimate $\|g\|_2/\|g\|_1$. Generally

$$\frac{\|g\|_2}{\|g\|_1} = \frac{(\sum_{\text{cosets } H_\gamma} (\sum_{m \in H_\gamma} \exp(-\|m^*/N\|_2^2))^{1/2}}{\sum_{m \in H_0} \exp(-\|m^*/N\|_2^2)}.$$

When $b_{k+1}, \dots, b_k$ form an independent subset of Γ it is easy to check that $H_0=0$. Therefore

$$\|g\|_2/\|g\|_1 = \left(\sum_{n \in \Gamma'} \exp(-2\|n^*/N\|_2^2) \right)^{1/2} = ((\beta_1 \cdots \beta_k)(1 + (\pi/2)^{1/2} N)^{s-k})^{1/2}$$

and we have the conclusion.

REMARK. In certain special cases like $s=2$ and $\Gamma=Z$, even when $H_0 \neq 0$ explicit computations show that $\|g\|_2/\|g\|_1 = O(N^{s/2})$. However, we are unable to establish this for the general case.

THEOREM 2. Let $b_1, \dots, b_k \in \Gamma$ have finite orders $\beta_1, \dots, \beta_k$ and $b_{k+1}, \dots, b_s \in \Gamma$ have infinite order. Let $E \subset \Gamma$ be a $\Lambda(p)$ set B as in the definition of $\Lambda(p)$ set. If $1 \leq r < \infty$ then

$$|A_r(N, s, b) \cap E| \leq B^2 e^2 (\beta_1 \cdots \beta_k)^{2/p} (1 + N^r)^{2(s-k)/p}.$$

PROOF. Again without loss of generality we take $b_0=0$. Consider first the case $r=1$. Define $g_{b_1}, \dots, g_{b_k}$ as in the proof of Theorem 1, and for $j=k+1, \dots, s$ define

$$g_{b_j}(x) = \sum_{n \in Z} \exp(-\|n/N\|_1) \langle x, nb_j \rangle.$$

Let $g(x) = \prod_{j=1}^s g_{b_j}(x)$ then $\hat{g}(\gamma)/\hat{g}(0) > e$ for all $\gamma \in A_1$ and, therefore, $|A_1 \cap E| \leq B^2 e^2 (\|g\|_2 / \|g\|_1)^{4/p}$. Let n_γ be any one element in the coset H_γ ; then as before

$$\begin{aligned} \|g\|_2 &= \left(\sum_{\text{cosets } H_\gamma} \left(\sum_{m \in H_0} \exp(-\|(m^* + n_\gamma^*)/N\|_1) \right)^2 \right)^{1/2} \\ &\leq \left(\sum_{\text{cosets } H_\gamma} \exp(-2 \|n_\gamma^*/N\|_1) \left(\sum_{m \in H_0} \exp(-\|m^*/N\|_1) \right)^2 \right)^{1/2}. \end{aligned}$$

Since $\|g\|_1 = \hat{g}(0) = \sum_{m \in H_0} \exp(-\|m^*/N\|_1)$ we have

$$\begin{aligned} \|g\|_2 / \|g\|_1 &\leq \left(\sum_{\text{cosets } H_\gamma} \exp(-2 \|n_\gamma^*/N\|_1) \right)^{1/2} \\ &\leq \left(\sum_{n \in \Gamma'} \exp(-2 \|n^*/N\|_1) \right)^{1/2} ((\beta_1 \cdots \beta_k)(1 + N)^{s-k})^{1/2}, \end{aligned}$$

and the conclusion follows for $r=1$. Now suppose $1 < r < \infty$. The trivial inequality $\sum_{j=k+1}^s |n_j| \leq \sum_{j=k+1}^s |n_j|^r$ shows that $A_r(N, s, \mathbf{b}) \subset A_1(N^r, s, \mathbf{b})$ and the conclusion follows for the general case.

COROLLARY. Suppose $E \subset \Gamma$ is a Λ set, then

$$|A_r \cap E| \leq 2C^2 e^3 \log((\beta_1 \cdots \beta_k)(1 + N^r)^{s-k})$$

where C is the constant in the definition of Λ set.

PROOF. In Theorem 2 set $p = 2 \log((\beta_1 \cdots \beta_k)(1 + N^r)^{s-k})$.

Kahane [2, p. 58] has already proven this result for the case where E is a Sidon set, and it is unknown if this condition is sufficient to make a set Sidon. The next result shows that in certain cases it is and therefore gives equality of Sidon and Λ sets (cf. M. P. Malliavin-Brammeret and P. Malliavin [3]).

COROLLARY. Let $E \subset \Gamma$ and suppose that each $\gamma \in \Gamma$ has prime order β . Then the following are equivalent.

- (i) $|A_1 \cap E| \leq Ks$ (K depends only on E).
- (ii) E is a Sidon set.
- (iii) E is a Λ set.

PROOF. We know that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Therefore, we show (i) $\Rightarrow$ (ii). The following result is due to Horn [1].

Let V be a vector space over some field and $E \subset V$. Given a natural number K , the following two statements are equivalent.

- (a) E is the union of K linearly independent sets.
- (b) $|B| \leq K \text{ rank}(B)$ for all finite subsets $B \subset E$, where $\text{rank}(B)$ is defined as the number of elements in a maximal linearly independent subset of B .

Γ is a vector space over $Z(\beta)$, and (i) implies that statement (b) of Horn's theorem is satisfied. Thus E is the union of K linearly independent sets and is, therefore, a Sidon set [4, p. 124].

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