COARSE UNIFORMITIES ON THE RATIONALS¹

H. C. ENOS

ABSTRACT. There exist uniform spaces μQ homeomorphic with the space Q of rational numbers such that every homeomorphic uniform space admits a uniformly continuous homeomorphism upon μQ . If "homeomorphism" is replaced by "bijection", the resulting weaker property is equivalent to having as completion a Peano continuum. With "homeomorphism", a (countable) dense subspace of a Hilbert cube has the property, but not a dense subspace of an interval.

Introduction. One can, in a sense, complete these results by describing all uniformities μ on Q such that every homeomorphic uniform space admits a uniformly continuous homeomorphism upon μQ as follows: they are those whose completion is a quotient space of a Hilbert cube by a mapping under which the inverse image of each point of Q is a single point. But nothing is known about such quotient spaces except that (of course) they are Peano continua, and their local cut points form a set of the first category.

The results extend investigations of Mrowka [3], who showed among other things that a countable metric space admits a uniformly continuous homeomorphism into a Hilbert cube. Mrowka asked whether a countable dense-in-itself subspace of an interval I can always be mapped bijectively to a dense subspace of I by a continuous mapping $I \rightarrow I$. It can, by the main result here on uniformly continuous bijections (Theorem 3 below, using also Theorem 1; we get the mapping by way of a Hilbert cube). Mrowka asked also whether the plane admits a continuous mapping into I whose restriction to a dense subset is homeomorphic. The result on local cut points (Theorem 2) shows that it does not.

I am indebted to the referee for replacing part of my proof with a reference to Fort [1].

1. Compatible case. Each uniformity μ on Q (or any other space) contains a precompact uniformity ν , still compatible with the topology.

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In turn ν contains a compatible metric uniformity π [2, p. 106], which is still precompact. So π is induced by an embedding of Q in a Hilbert cube H .

LEMMA. *Every subset of H homeomorphic with Q admits a uniformly continuous dense topological embedding into H .*

PROOF. The peculiar property of H that we need is this: each point has a basis of closed neighborhoods, homeomorphic with H , which admit continuous retractions upon their boundaries. That is evident for a geometric extreme point. It is true for all points since H is homogeneous [1].

Call the given subset $Q = \{q_1, q_2, \dots\}$. Its closure C is a perfect metric space and therefore admits a continuous mapping f_0 onto H . We construct mappings $f_n: C \rightarrow H$ such that the first n points q_i of Q are inverse images $f_n^{-1}f_n(q_i)$. Having f_n , the point $p_n = f_n(q_{n+1})$ differs from the previous $f_n(q_i)$. Let $\{N_{nj}\}$ be a sequence of neighborhoods of p_n , shrinking to p_n , admitting continuous retractions r_{nj} upon their boundaries. We want N_{n1} small, but we delay (till (α) and (β) below) specifying how small. Choose a neighborhood M_1 of q_{n+1} remote from $q_1, \dots, q_n$ and interior to $f_n^{-1}(N_{n1})$. The first modification s_1 of f_n is to satisfy the following conditions. On the complement of $f_n^{-1}(N_{n1})$, s_1 agrees with f_n ; on $f_n^{-1}(N_{n1}) - M_1$, s_1 is $r_{n1}f_n$; $s_1(M_1) = N_{n1}$, and $s_1(q_{n+1}) = p_n$. Such a continuous mapping exists because N_{n1} is an injective compact space and C is perfect. Observe that s_1 is onto; for the image contains N_{n1} , and s_1 agrees with f_n outside $f_n^{-1}(N_{n1})$. Then for a basis of neighborhoods $M_j \subset s_{j-1}^{-1}(N_{nj})$, we can construct s_j similarly. Since successive s_j differ only within the shrinking sets N_{nj} , they converge uniformly to a continuous surjection f_{n+1} , and $f_{n+1}^{-1}(p_n) = q_{n+1}$. To secure that the f_n converge uniformly, it suffices to require that (α) the sum of the diameters of all N_{n1} is finite. Then $g = \lim f_n$ is a continuous surjection. Now the early q_i , $i \leq n$, were $f_n^{-1}(p_{i-1})$. If (β) the diameters of the sets N_{n1} are sufficiently small fractions of the distance from p_n to $\{p_0, \dots, p_{n-1}\}$, then we get $q_i = g^{-1}(p_{i-1})$. Since C is compact, g is uniformly continuous. Since g is surjective, $g(Q)$ is dense. Ultrafilters in Q having no limit in Q have limits in $C - Q$, and g takes them to nonconvergent ultrafilters in $g(Q)$; so $g|Q$ is a topological embedding.

M. K. Fort showed [1] that any two countable dense subsets D, E of H are equivalent by an autohomeomorphism of H . Thus there is, up to isomorphism, a unique uniform space ρQ which is homeomorphic with Q and has completion homeomorphic with H . We have proved:

THEOREM 1. *Every uniform space homeomorphic with Q admits a uniformly continuous homeomorphism upon ρQ .*

Shifting away from statements "up to isomorphism", Q has many uniformities ρ with completion H , and the result is that for each uniformity μ there is a coarser ρ . What other equivalence classes (under autohomeomorphism) of uniformities σ have that property? Evidently, those which are coarser than some ρ . Such a uniformity σ is again precompact. The completion of σQ is a quotient space of H by a decomposition in which each single point of Q is an equivalence class. Conversely, the uniformity induced by such a quotient space is coarser than some ρ .

THEOREM 2. *For Peano continua, the property that the set of local cut points is of the first category is preserved by continuous surjections whose restriction to a dense set is homeomorphic.*

PROOF. Let $f: K \rightarrow L$ be such a mapping. L is also a Peano continuum. By Lavrentiev's theorem, f maps a dense G_δ set D homeomorphically. Removing a first category F_σ set containing the local cut points, we have a dense G_δ set E of nonlocal cut points mapped homeomorphically. Since E is dense and K compact, E is $f^{-1}f(E)$, and $L - f(E)$ is first category. In L , a connected neighborhood of a point $f(e)$ cannot be partitioned into $f(e)$ and two open sets V_1, V_2 . For the inverse image would partition accordingly into e, U_1, U_2 . Then e is not in the closures of both U_i . But $f(U_i^-)$ is compact and must contain $f(e)$.

Evidently a decomposition required to be homeomorphic on a dense set can do almost anything on a first category set. For instance, for $Q \subset H$, one can find a disjoint sequence of boundaries of neighborhoods of its points, with diameters converging to 0; this yields a decomposition of H homeomorphic on Q with the quotient one-dimensional at each point of Q .

2. Incompatible case. A separated uniformity on Q , compatible with the topology or not, but coarser than ρ , must be precompact; the completion is a quotient of H by a mapping injective on Q . Since it is a quotient of H , it is a Peano continuum. Conversely:

THEOREM 3. *For every countable dense set D in a Peano continuum K there is a continuous map $H \rightarrow K$ taking Q bijectively to D .*

PROOF. First, there is a continuous surjection $f_0: I \rightarrow K$. Since K is locally arcwise connected and D is dense in K , a sequence of small modifications (f_n to f_{n+1}) will converge to $g: I \rightarrow K$ such that $g^{-1}(D)$ is dense. Keeping each f_n surjective, g will be surjective. One can secure also that g is not constant on any interval. Then the inverse image of any finite set is nowhere dense; hence $g^{-1}(D)$ contains a dense set on which g is injective, and a larger dense set A mapped bijectively to D .

Evidently two countable dense sets in I containing the same number of end points are equivalent by a homeomorphism of I . Mrowka showed [3]

that there is a map $H \rightarrow I$ mapping $Q \subset H$ injectively. It is easy to see that one can prescribe the number of endpoint values; and the image of Q must be dense.

THEOREM 4. *A countable uniform space is a bijective uniformly continuous image of every countable dense-in-itself uniform space if and only if its completion is a Peano continuum.*

PROOF. As remarked before Theorem 3, a bijective uniformly continuous image of ρQ has a Peano continuum for completion. Given such a space λD with completion K , given a countable dense-in-itself space μX , evidently μ contains a separated metric uniformity ν . Since μX is dense-in-itself, so is νX ; thus it is homeomorphic with Q . Then Theorem 1 and Theorem 3 yield a uniformly continuous bijection $\nu X \rightarrow \lambda D$.

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DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO,
AMHERST, NEW YORK 14226