A NORMAL FORM IN THE HOMEOTOPY GROUP OF A SURFACE OF GENUS 2, WITH APPLICATIONS TO 3-MANIFOLDS

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ABSTRACT. It is shown that elements in the homeotopy group of a closed, compact, orientable 2-manifold of genus 2 can be put into a unique normal form which allows them to be enumerated systematically. As an application, the class of 3-manifolds which admit Heegaard splittings of genus 2 are shown to be denumerable, and a procedure is given for enumerating presentations for their fundamental groups.

1. Introduction. Let T be a closed, compact orientable 2-manifold of genus 2, and let H be its homeotopy group, that is, the group of all homeomorphisms of $T \rightarrow T$ modulo the subgroup of those homeomorphisms which are isotopic to the identity. Generators and defining relations for the group H (or rather a subgroup of index 2 in H) were determined in [1]. We will show here that the elements in H can be put into a unique normal form which shows that they are denumerable, and we will give an algorithm for enumerating them systematically.

This result has potential applications in the study of 3-manifolds. Define a "genus 2 Heegaard splitting" to be a compact 3-manifold M without boundary which admits a decomposition as $M = X \cup X'$, where X and X' are solid handle-bodies of genus 2, and $X \cap X' = \text{bd } X = \text{bd } X'$. Each such splitting can be associated (nonuniquely) with an element in H , so that our enumeration of the elements in H is equivalent to an enumeration (with duplications) of all possible genus 2 Heegaard splittings. We show how to use our enumeration to enumerate presentations (in terms of 2-generators, 2 relators) of the fundamental groups of all genus 2 Heegaard splittings. This enumeration might be useful as a tool in studying the classification of 3-manifolds.

At the conclusion we discuss briefly why the approach used here does not generalize to genus > 2 .

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2. Enumerating the elements in H . Our enumeration will be based on the following reasoning: we will first show that the structure of H is closely related to the structure of the 5-string Artin braid group, B_5 . We will then use results due to F. Garside [3] on the braid group to develop a normal form in H .

Instead of H , we first consider the mapping class group M of the surface T , where M is defined to be the subgroup of H in which admissible maps are restricted to those homeomorphisms of $T \rightarrow T$ which preserve orientation. Using the well-known fact that the group H is isomorphic to $\text{Aut } \pi_1 T / \text{Inn } \pi_1 T$, we can interpret elements of H and M as classes of automorphisms of the fundamental group $\pi_1 T$ of the surface T . Let $\pi_1 T$ be presented in terms of canonical generators a_1, b_1, a_2, b_2 which satisfy the single relation $[a_1, b_1][a_2, b_2] = 1$. Let $\tau_1, \dots, \tau_5$ be the automorphism classes of:

$$\begin{aligned}
 (1) \quad & \tau_1: a_1 \rightarrow a_1 b_1^{-1}, \\
 & \tau_2: b_1 \rightarrow b_1 a_1, \\
 & \tau_3: a_1 \rightarrow a_1 b_1^{-1} a_2 b_2 a_2^{-1}, \\
 & \quad a_2 \rightarrow a_2 b_2^{-1} a_2^{-1} b_1 a_2, \\
 & \quad b_1 \rightarrow a_2 b_2^{-1} a_2^{-1} b_1 a_2 b_2 a_2^{-1}, \\
 & \tau_4: b_2 \rightarrow b_2 a_2, \\
 & \tau_5: a_2 \rightarrow a_2 b_2^{-1},
 \end{aligned}$$

where all generators which are not listed explicitly are assumed to be kept fixed. Then M admits the presentation [1]:

generators: $\tau_1, \dots, \tau_5$,

relations:

$$(2) \quad [\tau_i, \tau_j] = 1 \quad \text{if } |i - j| \geq 2, \quad 1 \leq i, j \leq 5,$$

$$(3) \quad \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \quad 1 \leq i \leq 4,$$

$$(4) \quad (\tau_1 \tau_2 \tau_3 \tau_4 \tau_5)^6 = 1,$$

$$(5) \quad (\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1)^2 = 1,$$

$$(6) \quad [\tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1, \tau_i] = 1, \quad 1 \leq i \leq 5.$$

The group H is an extension of M by a cyclic group of order 2. Define a new element $\tau_6 \in H$ by the rule:

$$\begin{aligned}
 (7) \quad & \tau_6: a_1 \rightarrow b_1, \\
 & \quad a_2 \rightarrow b_2, \\
 & \quad b_1 \rightarrow a_1, \\
 & \quad b_2 \rightarrow a_2.
 \end{aligned}$$

Since τ_6 is orientation-reversing, the pair $(1, \tau_6)$ can be used as coset representatives for M in H . Thus if we can show that the elements in M are denumerable, it will follow that those in H are also denumerable, with each element $x \in M$ corresponding to two elements x and $x\tau_6$ in H .

To develop a normal form for elements in M , we first observe that it follows from relations (2)–(6) that the element $\alpha = \tau_1\tau_2\tau_3\tau_4\tau_5^2\tau_4\tau_3\tau_2\tau_1$ is in the center of M and has order 2. Let G be the quotient group obtained by setting $\alpha=1$, and let $\rho_i \in G$ be the image of $\tau_i \in M$ under the homomorphism from $M \rightarrow G$. Then G is generated by $\rho_1, \dots, \rho_5$ and has defining relations $(2)_\rho, (3)_\rho, (4)_\rho$ and:

$$(8) \quad \rho_1\rho_2\rho_3\rho_4\rho_5^2\rho_4\rho_3\rho_2\rho_1 = 1$$

where $(2)_\rho, (3)_\rho, (4)_\rho$ mean relations (2), (3), (4) but with the ρ 's substituted for the τ 's. If we could list elements in G , we could list elements in M , by lifting the element $\rho_{\mu_1}^{\epsilon_1} \cdots \rho_{\mu_r}^{\epsilon_r}$ in G to the pair $\tau_{\mu_1}^{\epsilon_1} \cdots \tau_{\mu_r}^{\epsilon_r}$ and $\alpha\tau_{\mu_1}^{\epsilon_1} \cdots \tau_{\mu_r}^{\epsilon_r}$ in M .

Now we note that if we add to relations $(2)_\rho, (3)_\rho, (4)_\rho$ and (8) the additional relation that the square of every generator is trivial, then our presentation for G will go over to a presentation for the symmetric group Σ_6 on 6 letters [2, equation 6.28], each ρ_i being mapped onto the transposition $(i, i+1)$. Let K be the subgroup of G consisting of all elements in G whose images in Σ_6 leave the letter 6 invariant. Then K has index 6 in G , and each element $\beta \in G$ has a unique representation $\beta = \gamma k_\mu$, where $\gamma \in K$ and the elements $k_0, \dots, k_5$ are coset representatives for K in G , with

$$k_0 = 1, \quad k_\mu = \rho_5\rho_4 \cdots \rho_\mu \quad \text{if } \mu = 1, \dots, 5.$$

Thus our enumeration is reduced to an enumeration in K . Since the elements $k_0, k_1, \dots, k_5$ are a Schreier set of coset representatives for K in G , we can apply the Schreier-Reidemeister method [4] to the known presentation for G to obtain a presentation for K . This calculation gives that K is generated by $\rho_1, \rho_2, \rho_3, \rho_4$ and has defining relations:

$$(9) \quad \rho_i\rho_j = \rho_j\rho_i, \quad 1 \leq i, j \leq 4, \quad |i-j| \geq 2,$$

$$(10) \quad \rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}, \quad 1 \leq i \leq 4,$$

$$(11) \quad (\rho_1\rho_2\rho_3\rho_4)^5 = 1.$$

To understand the group K , we compare it to a group which is closely related to it, the 5-string Artin braid group, B_5 . The n -string braid group B_n can be defined as the abstract group with generators $\sigma_1, \dots, \sigma_{n-1}$ and defining relations $(9)_\sigma$ and $(10)_\sigma$ where the indices i, j are allowed to run between 1 and $n-1$ instead of between 1 and 4. Now, the element

$(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$ generates the center $Z(B_n)$ of B_n (see, for example, [3]). Hence every element $\delta \in B_n$ can be written uniquely in the form

$$\delta = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{nm} V(\sigma_1, \cdots, \sigma_{n-1})$$

where m is an integer. Under the natural homomorphism from $B_5 \rightarrow K$ the element $(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^{5m} V(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ in B_5 goes over to the element $V(\rho_1, \rho_2, \rho_3, \rho_4)$ in K , where two words V and V' in K will be equivalent iff their preimages in B_5 are equivalent for some fixed m . Thus if we can develop a scheme for enumerating the distinct coset representatives of the center $Z(B_5)$ in B_5 , we will also be able to enumerate the distinct elements of K , which will in turn enable us to enumerate elements in G , in M , and finally in H .

The braid group B_n was studied by Garside in [3]. Garside developed a normal form for words in B_n , which we claim allows us to enumerate the coset representatives of $Z(B_n)$. In B_n , define the elements:

$$(12) \quad \Delta = (\sigma_1 \cdots \sigma_{n-1})(\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1 \sigma_2)(\sigma_1), \quad \phi = (\sigma_1 \cdots \sigma_{n-1})^n.$$

The element ϕ generates $Z(B_n)$. Moreover, as a consequence of relations (9) _{σ} and (10) _{σ} one can show that in B_n :

$$(13) \quad \Delta^2 = \phi.$$

Now, Garside finds coset representatives for the infinite cyclic subgroup of B_n generated by Δ . If we denote his coset representatives by $\{P_i, i \in I\}$, then it is immediate that $\{P_i, \Delta P_i; i \in I\}$ will be a set of coset representatives for $Z(B_n)$. So we need only show that Garside's coset representatives are denumerable.

The words P_i obtained by Garside have a very special form which we describe briefly. Let S_n be the semigroup with generators $\sigma_1, \cdots, \sigma_{n-1}$ and with the same defining relations as B_n . Garside shows that S_n can be imbedded in B_n , and that the coset representatives $\{P_i, i \in I\}$ can be chosen to be in S_n , i.e. so that they involve only positive powers of the σ_i . He also gives a procedure for testing whether a given $P(\sigma_1, \cdots, \sigma_{n-1}) \in S_n$ is a coset representative (i.e. is "prime to Δ " and is "lexicographically smallest" in the "diagram of P " in Garside's notation; for details see [3]). Garside's procedure for finding the coset representative of a given word goes over immediately to a procedure for listing coset representatives: one simply lists all words in S_n in order of increasing length, finds the coset representative of each such word, and enters it in the list if it is not a coset representative which has appeared earlier on the list.

It then follows that the elements of our original group H are also denumerable, because each coset representative $P(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ of the

cyclic subgroup $\langle \Delta \rangle$ in B_5 corresponds to precisely 48 distinct words $W(\tau_1, \dots, \tau_6) \in H$, i.e., the words:

$$W(\tau_1, \dots, \tau_6) = \alpha^{\varepsilon_1} \Delta^{\varepsilon_2} P(\tau_1, \tau_2, \tau_3, \tau_4) k_\mu \tau_6^{\varepsilon_3}$$

where:

$$\alpha = \tau_1 \tau_2 \tau_3 \tau_4 \tau_5^2 \tau_4 \tau_3 \tau_2 \tau_1, \quad \Delta = \tau_1 \tau_2 \tau_3 \tau_4 \tau_1 \tau_2 \tau_3 \tau_1 \tau_2 \tau_1,$$

$$\mu = 0, 1, 2, 3, 4, 5,$$

$$k_\mu = \tau_5 \tau_4 \cdots \tau_\mu \quad \text{if } \mu = 1, 2, 3, 4, 5,$$

$$= 1 \quad \text{if } \mu = 0,$$

$$\varepsilon_1, \varepsilon_2, \varepsilon_3 = 0 \quad \text{or } 1.$$

3. Enumerating the fundamental groups of the genus 2 Heegaard splittings. Let $\tau \in H$, and suppose that τ is represented by an automorphism which acts on $\pi_1 T$ as follows:

$$\tau: a_i \rightarrow A_i(a_1, a_2, b_1, b_2),$$

$$b_i \rightarrow B_i(a_1, a_2, b_1, b_2), \quad i = 1, 2.$$

Then it is shown in [5, Theorem 10.1] that a genus 2 Heegaard splitting exists with fundamental group:

$$(14) \quad \pi_1(X \cup X') = \langle a_1, a_2; A_i(a_1, a_2, 1, 1), i = 1, 2 \rangle.$$

The enumeration procedure described in §2 can be used to obtain an enumeration of the relators $A_i(a_1, a_2, 1, 1)$, $i=1, 2$, by using equation (1) and (7) to compute the action of τ on the elements $a_1, a_2 \in \pi_1 T$ for each $\tau \in H$. Thus we could enumerate the fundamental groups of all 3-manifolds which admit genus 2 Heegaard splittings. Of course there will be duplications, both because the same words A_1 and A_2 will appear repeatedly, and more seriously because no procedure exists for distinguishing isomorphism classes of groups which are defined by presentations.

4. Generalizing to genus >2 . The analysis given above does not carry over to genus >2 , because it depends crucially on the presentation obtained in [1] for M , which generalizes to a presentation for a *proper subgroup* of the mapping class group $M(T_g)$ of a closed, compact orientable surface T_g of genus $g > 2$. The algebraic relationship between this subgroup and the full group $M(T_g)$ is not, to the author's knowledge, well understood.

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