REPRESENTATION OF THE PROJECTABLE AND STRONGLY PROJECTABLE HULLS OF A LATTICE-ORDERED GROUP¹

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ABSTRACT. A representable l -group G can be embedded into a projectable (strongly projectable) l -group; an essential extension of G which is minimal with respect to being projectable (strongly projectable) is unique. In this paper these projectable and strongly projectable "hulls" of G are represented using direct limits. If G is an f -ring, or f -ring without nonzero nilpotent elements, then so are these hulls.

1. Introduction. An l -group G is said to be *representable* if it can be embedded into a cardinal product of totally ordered groups, *projectable* if each polar subgroup of the form g'' for $g \in G$ is a cardinal summand of G , and *strongly projectable* if every polar subgroup of G is a cardinal summand. If G is projectable then each polar subgroup of G is a normal subgroup and so G , and hence each l -subgroup of G , is representable [4, Theorem 1.8]. Conversely, if G is representable then G is a dense l -subgroup of its orthocompletion [2, pp. 116, 125] which is strongly projectable.

If G is a large l -subgroup of a strongly projectable (projectable) l -group K then the intersection H of all strongly projectable (projectable) l -subgroups of K containing G is strongly projectable (projectable), and Conrad [5] has shown that H is the unique (up to l -isomorphism extending the identity map of G) minimal strongly projectable (projectable) essential extension of G , and called H the *sp-hull* (*p-hull*) of G . In this note we wish to give direct limit representations of these hulls in a fashion similar to that in which Conrad [3, pp. 455–457] has represented the orthocompletion of G . References containing related material are [1], [2], [5], [7].

NOTATION. We use the notation and terminology of the general references [4], [6] unless otherwise specified. G will always denote a

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representable l -group (written additively, but not assumed to be abelian). For a subset S of G , S' denotes the polar of S , and $\mathcal{P}(G)$ the complete Boolean algebra of polar subgroups of G . $\prod A_\lambda$ denotes the cardinal product of the l -groups A_λ and $A \oplus B$ denotes the cardinal sum of the l -groups A, B .

2. Representation of the hulls. The following discussion of the direct limit of a directed system of l -groups and l -isomorphisms and construction of the orthocompletion of G was given by Conrad in [3], but is repeated here for completeness.

Let Λ be a lower directed set and suppose $\{G_\lambda, \Pi_{\alpha\beta}, \Lambda\}$ is a lower directed system of l -groups and l -isomorphisms (i.e., if $\alpha \geq \beta$ then $\Pi_{\alpha\beta}$ is an l -isomorphism of G_α into G_β where the usual transitive laws are satisfied). If P is the set product of the G_λ , let L denote the collection of all $t \in P$ such that if $\alpha \geq \beta$ then $t_\alpha \neq 0$ or $t_\beta = 0$ implies $t_\alpha \Pi_{\alpha\beta} = t_\beta$, and $t_\alpha = 0$ and $t_\beta \neq 0$ implies $t_\beta \notin G_\alpha \Pi_{\alpha\beta}$.

Let θ denote the 0 element of P , and for $\theta \neq k$, $t \in L$ select α such that $k_\alpha \neq 0 \neq t_\alpha$; then $k+t$ is defined by $(k+t)_\alpha = k_\alpha + t_\alpha$ (each nonzero component of a nonzero element of L completely determines that element). Also call $t \in L$ *positive* if $t = \theta$ or $t_\alpha > 0$ for some α . Then L is an l -group and the direct limit of the directed system above; if $t \in L$ and t_α is not comparable with 0 then $(t \vee \theta)_\alpha = t_\alpha \vee 0$.

Call a maximal disjoint subset of $\mathcal{P}(G)$ a *partition* and let $D(G)$ denote the set of all partitions of $\mathcal{P}(G)$. If $\mathcal{A}$ and $\mathcal{C}$ are partitions then we write $\mathcal{A} \leq \mathcal{C}$ ($\mathcal{A}$ *refines* $\mathcal{C}$) if $A \in \mathcal{A}$ implies $A \subseteq C$ for some $C \in \mathcal{C}$. If $\mathcal{B} \in D(G)$, let $\mathcal{D} = \{C \cap B \neq \emptyset \mid C \in \mathcal{C}, B \in \mathcal{B}\}$ (the *intersection* of $\mathcal{B}$ and $\mathcal{C}$); then $\mathcal{D} \leq \mathcal{B}, \mathcal{C}$ and so $D(G)$ is a lower directed set. If $C \in \mathcal{C}$ and $\{A_\gamma\}_\Gamma$ is the collection of all elements of $\mathcal{A}$ included in C then $C' = \bigcap A'_\gamma$ and so the natural map of G/C' into $\prod \{G/A'_\gamma \mid \gamma \in \Gamma\}$ is an l -isomorphism. In this way a natural l -isomorphism $\Pi_{\mathcal{C}, \mathcal{A}}$ of $G_{\mathcal{C}} = \prod \{G/C' \mid C \in \mathcal{C}\}$ into $G_{\mathcal{A}}$ is determined, and $\{G_{\mathcal{C}}, \Pi_{\mathcal{C}, \mathcal{A}}, D(G)\}$ is then a directed system; let its direct limit be L . If $t \in L$ and $\mathcal{C} \in D(G)$ we write $t_{\mathcal{C}} = (\dots, t(C) + C', \dots)$ where $C \in \mathcal{C}$ and $t(C) \in G$ (the $t(C)$ are only determined mod C' , of course). If $g \in G$ and $\mathcal{B} \in D(G)$ let $\tilde{g}(B) = g$ for all $B \in \mathcal{B}$; then the map $g \rightarrow \tilde{g}$ is an l -isomorphism of G onto a dense l -subgroup $\tilde{G}$ of L . L is the orthocompletion of G in the sense of Bernau [2] (see [3, pp. 455–457]).

Now to construct the sp-hull of G let $F(G)$ denote the set of all finite elements of $D(G)$ and consider the directed system $\{G_{\mathcal{C}}, \Pi_{\mathcal{C}, \mathcal{A}}, F(G)\}$. If S is the direct limit of this system then we have a natural l -isomorphism $g \rightarrow \tilde{g}$ of G into S given by $\tilde{g}(B) = g$ for all $B \in \mathcal{B}$ and all $\mathcal{B} \in F(G)$, and $\tilde{G}$ is a dense l -subgroup of S . Therefore the map $Q \rightarrow Q \cap \tilde{G}$ determines a natural Boolean isomorphism of $\mathcal{P}(S)$ onto $\mathcal{P}(G)$ [3, p. 455], and so if

$P \in \mathcal{P}(G)$ and $*$ denotes the polar operation in S , then $\tilde{G} \cap \tilde{P}^{**} = \tilde{P}$ (henceforth we will identify G and $\tilde{G}$).

If $\theta \neq f \in S$, $P \in \mathcal{P}(G)$, and $\mathcal{B} \in F(G)$ refines $\{P, P'\}$ then $f \in P^*$ if and only if $f(B) = 0 \bmod B'$ for all $B \in \mathcal{B}$ included in P . Similarly, $f \in P^{**}$ if and only if $f(B) = 0 \bmod B'$ for each $B \in \mathcal{B}$ included in P' . The proofs of these facts are routine.

THEOREM A. S is the *sp-hull* of G .

PROOF. Let $Q \in \mathcal{P}(S)$ and $\theta \neq t \in S$; Q is of the form P^{**} for a unique $P \in \mathcal{P}(G)$. Pick $\mathcal{B} \in F(G)$ such that $\mathcal{B}$ refines $\{P, P'\}$ and $t_{\mathcal{B}} \neq 0$, and define $k \in S$ by specifying that, for all $B \in \mathcal{B}$, $k(B) = t(B)$ if $B \subseteq P$, and $k(B) = 0$ otherwise. Then $k \in P^{**} = Q$ and $t - k \in P^* = Q^*$, so that $t = k + (t - k) \in Q \oplus Q^*$. Therefore S is a strongly projectable essential extension of G .

Now suppose that $G \subseteq H \subseteq S$ where H is strongly projectable and let $\perp$ denote the polar operation in H . Assume that $t \in H$, $Q \in \mathcal{P}(H)$, and write $t = t_1 + t_2$ in $Q \oplus Q^\perp = H$ and $t = k_1 + k_2$ in $Q^{**} \oplus Q^* = S$. Since $Q \subseteq Q^{**}$ and $Q^\perp \subseteq Q^*$ it follows that $t_1 = k_1$ and $t_2 = k_2$; thus the projections t_1 of t into Q in H and k_1 of t into Q^{**} in S are identical.

Now let $\theta \neq t \in S$ with $0 \neq t_{\mathcal{C}} = (g_1 + C'_1, \dots, g_n + C'_n)$ and let $t_j \in S$ be determined by $t_j(C_j) = g_j$ and $t_j(C_i) = 0$ for $i \neq j$, $1 \leq j \leq n$. Then t_j is the projection of g_j into C_j^{**} , and since $g_j \in H$ it follows as discussed in the last paragraph that t_j is also the projection of g_j into $C_j^{\perp\perp}$. Thus $t_j \in H$ and so $t = t_1 + \dots + t_n \in H$. Hence $H = S$, and so S is a minimal strongly projectable essential extension of G .

REMARK. If A is an f -ring then each polar subgroup of A is an ideal and so any directed system of the $A_{\mathcal{C}}$ and $\Pi_{\mathcal{C}, \mathcal{A}}$ is a system of f -rings and l -ring isomorphisms. Thus the direct limit of such a system is also an f -ring. If A has no nonzero nilpotent elements then A/P also has none for each polar subgroup P of A , and hence such a direct limit also has this property.

In particular, the *sp-hull* S of A is an f -ring (with no nonzero nilpotent elements if A has none), and the map $g \rightarrow \tilde{g}$ is an l -ring isomorphism of A into S .

The *p-hull* of G can be constructed in an analogous (but somewhat more complicated) fashion. Here let T denote the sublattice of $\mathcal{P}(G)$ generated by all polars of the form g'' or g' for $g \in G$ and let $T(G)$ be the set of all $\mathcal{C} \in F(G)$ which are subsets of T . This gives rise to a directed system $\{G_{\mathcal{C}}, \Pi_{\mathcal{C}, \mathcal{A}}, T(G)\}$, the direct limit of which we denote by T . Again we have a natural embedding $g \rightarrow \tilde{g}$ given by $\tilde{g}(C) = g$ for all $C \in \mathcal{C}$ and all $\mathcal{C} \in T(G)$ and we denote the polar operations in G and T by $'$ and $*$ respectively. We identify G and $\tilde{G}$.

LEMMA. If $\theta \neq k \in T$ with $0 \neq k_{\mathcal{C}} = (\cdots, k_j + C'_j, \cdots)$ then $k^{**} = \bigvee (C_j^{**} \cap k_j^{**})$ where the sup occurs in $\mathcal{P}(T)$.

PROOF. Without loss of generality $k > \theta$. Assume first that $\mathcal{C} = \{C_j\}_1^n$ and $0 \neq k_{\mathcal{C}} = (g + C'_1, 0, \cdots, 0)$. If $\mathcal{D}$ is the intersection of $\mathcal{C}$ and $\{g'', g'\}$ then $0 \neq k_{\mathcal{D}} = (g + (C_1 \cap g'')', 0, \cdots, 0)$ and so $k \in (C_1 \cap g'')^{**} = C_1^{**} \cap g^{**}$. We wish to show that if $\theta < t \in C_1^{**} \cap g^{**}$ then $t \wedge k > \theta$; then it will follow that $k^{**} = C_1^{**} \cap g^{**}$, as desired.

Consider $\theta < t \in C_1^{**} \cap g^{**}$ and let $t_{\mathcal{B}} > 0$ for some $\mathcal{B}$ refining $\mathcal{D}$. Then $t(B) = 0 \bmod B'$ when $B \in \mathcal{B}$ is not included in $C_1 \cap g''$, and if $B \subseteq C_1 \cap g''$ and $g \notin B'$ then $g + B'$ is a weak unit for G/B' . Therefore if $t(B), g \notin B'$ for some $B \in \mathcal{B}$ then $t_{\mathcal{B}} \wedge k_{\mathcal{B}} > 0$ and so $\theta < t \wedge k$ as desired. Suppose by way of contradiction that for all $B \in \mathcal{B}, g \notin B'$ implies $t(B) \in B'$. Then $g_{\mathcal{B}} \wedge t_{\mathcal{B}} = 0$ and so $t \in g^*$. Hence $\theta < t \in g^* \cap g^{**} = \theta$, a contradiction. It follows then that $t \wedge k > \theta$ and so $k^{**} = C_1^{**} \cap g^{**}$.

In the general case $\theta < k$ and $0 < k_{\mathcal{C}} = (k_1 + C'_1, \cdots, k_n + C'_n)$. If $\theta \leq t_j \in T$ is determined by $(t_j)_{\mathcal{C}} = (0, \cdots, 0, k_j + C'_j, 0, \cdots, 0)$, then

$$t_j^{**} = k_j^{**} \cap C_j^{**} \quad \text{and} \quad k = \bigvee t_j$$

so that $k^{**} = \bigvee t_j^{**} = \bigvee (k_j^{**} \cap C_j^{**})$.

THEOREM B. T is the p -hull of G .

PROOF. Suppose $\theta \neq k, f \in T$ and $0 \neq k_{\mathcal{C}} = (k_1 + C'_1, \cdots, k_n + C'_n)$. Then by the lemma $k^{**} = \bigvee (C_j^{**} \cap k_j^{**})$, and so if $k^{**} \cap G = P \in \mathcal{P}(G)$ then $P = \bigvee (G \cap C_j^{**} \cap k_j^{**}) = \bigvee (C_j \cap k_j'')$. Thus $P \in T$ and hence $\{P, P'\} \in T(G)$. Pick $\mathcal{B} \in T(G)$ refining $\{P, P'\}$ such that

$$0 \neq f_{\mathcal{B}} = (f_1 + B'_1, \cdots, f_m + B'_m),$$

and let $t \in T$ be specified by $t(B_j) = f_j$ if $B_j \subseteq P$ and $t(B_j) = 0$ otherwise. Then $f = t + (f - t) \in P^{**} \oplus P^* = k^{**} \oplus k^*$, and so it follows that T is projectable.

Now suppose that $G \subseteq H \subseteq T$ where H is projectable and let $\perp$ denote the polar operation in H . If $h \in H$ then $h^{\perp} \subseteq h^*$ and $h^{\perp\perp} \subseteq h^{\perp\perp**} = h^{**}$ [5, (5i)]; thus if $k \in H$ then the projections of k into $h^{\perp\perp}$ in H and into h^{**} in T coincide.

Assume now that $\theta < t \in T$ with $t_{\mathcal{C}} > 0$.

Case 1. Suppose first that $t_{\mathcal{C}} = (g + C'_1, 0, \cdots, 0)$ where without loss of generality $0 < g \notin C'_1$. Since $C_1 \in T$ we have $C_1 = \bigvee_I \bigwedge_J S_{ij}$ where I and J are finite sets and S_{ij} has the form g_{ij}'' or g_{ij}' for some $g_{ij} \in G$. Let $\mathcal{B}$ be the intersection of all the partitions $\{S_{ij}, S_{ij}'\}$; we claim that $\mathcal{B}$ refines $\{C_1, C'_1\}$. For if $B \in \mathcal{B}$ and $B \not\subseteq C_1$, then for each i there is a j_i such that $B \not\subseteq S_{ij_i}$. Then $B \subseteq S_{ij_i}'$ and so $B \subseteq \bigwedge_I \bigvee_J S_{ij}' = C'_1$.

Now let $t_{ij} \in T$ be determined by $t_{ij}(B) = g$ if $B \subseteq S_{ij}$ and $t_{ij}(B) = 0$ otherwise, and let $k = \bigvee_I \bigwedge_J t_{ij}$. If $k(B) \neq 0 \bmod B'$ then $B \subseteq C_1$, while if $B \subseteq C_1$ then $B \not\subseteq C'_1$ and so it follows that $B \subseteq \bigwedge_J S_{ij}$ for some i , and hence $k(B) = g = t(B)$. Therefore $t_{\mathcal{B}} = k_{\mathcal{B}}$ and so $t = \bigvee_I \bigwedge_J t_{ij}$. Since t_{ij} is the projection of $g \in H$ into $S_{ij}^{**} = g_{ij}^{**}$ or g_{ij}^* , we have $t_{ij} \in H$; thus $t \in H$.

Case 2. In the general case we have $t = t_1 + \cdots + t_n$ where the t_j have the form considered in Case 1. Thus $t_j \in H$ and so $t \in H$. Therefore $H = T$, and so T is a minimal projectable essential extension of G .

REMARK. It follows as before that if A is an f -ring, or an f -ring with no nonzero nilpotent elements, then so is the p -hull of A .

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