MINIMAL SURFACES WITH CONSTANT GAUSS CURVATURE

BANG-YEN CHEN¹

ABSTRACT. Minimal surfaces with constant Gauss curvature in real space forms are studied.

1. Introduction. Let $R^m(c)$ be an m -dimensional complete, simply-connected Riemannian manifold of constant curvature c with covariant differentiation ∇ , and let M be a surface immersed in $R^m(c)$. For a normal vector field η of M in $R^m(c)$, we denote by $\nabla^*\eta$ the normal component of $\nabla\eta$. Then ∇^* defines a connection in the normal bundle. A normal vector field η is said to be parallel in the normal bundle if $\nabla^*\eta=0$ and to be *constant* if $\nabla\eta=0$.

MAIN THEOREM. *Let M be a minimal surface with constant Gauss curvature in a space form $R^m(c)$ of curvature c . If there exists a nonconstant normal vector field which is parallel in the normal bundle, then $c>0$ and M is an open piece of the Clifford torus.*

Since every unit normal vector field is parallel in the normal bundle for $m=3$ and is constant only if M is totally geodesic, we have the following corollary.

COROLLARY 1. *If M is a minimal surface with constant Gauss curvature in a 3-space form $R^3(c)$, then either M is totally geodesic or $c>0$ and M is an open piece of the Clifford torus.*

REMARK 1. Corollary 1 was obtained by Lawson [4] for $c>0$. If $c=0$, then Corollary 1 also follows from the well-known Ricci condition.

REMARK 2. Suppose $R^m(c)$ is the euclidean m -space E^m . Then by combining a result of Osserman-Chern [5] for the generalized Gauss map and a result of Calabi [1] for Riemann surfaces in the complex projective spaces, we know that *every minimal surface in the euclidean m -space with constant Gauss curvature must be totally geodesic.*

Received by the editors October 1, 1971.

AMS 1970 subject classifications. Primary 53A10, 53B25; Secondary 53C40.

Key words and phrases. Minimal surface, Gauss curvature, scalar normal curvature, Clifford torus.

¹ This work was partially supported by the National Science Foundation under grant GU-2648.

© American Mathematical Society 1972

REMARK 3. For the minimal surfaces with positive constant Gauss curvature in the m -sphere, see do Carmo-Wallach [2].

The author would like to express his thanks to Professor H. B. Lawson for the valuable conversations.

2. Preliminaries. Let $R^m(c)$ be a complete, simply-connected Riemannian manifold of constant sectional curvature c , with scalar product $\langle \cdot, \cdot \rangle$, and let M be a surface immersed in $R^m(c)$. We choose a local field of orthonormal frames $e_1, \dots, e_m$ in $R^m(c)$ such that, restricted to M , the vectors e_1, e_2 are tangent to M (and, consequently, $e_3, \dots, e_m$ are normal to M). With respect to the frame field chosen above, let $\omega^1, \dots, \omega^m$ be the field of dual frame. Then the structure equations of $R^m(c)$ are given by

$$(1) \quad d\omega^A = -\sum \omega_B^A \wedge \omega^B, \quad \omega_B^A + \omega_A^B = 0, \\ A, B, C, \dots = 1, \dots, m.$$

$$(2) \quad d\omega_B^A = -\sum \omega_C^A \wedge \omega_B^C + c\omega^A \wedge \omega^B,$$

We restrict these forms to M . Then $\omega^r = 0$, for $r, s, t, \dots = 3, \dots, m$. Since $0 = d\omega^r = \omega_r^1 \wedge \omega^1 + \omega_r^2 \wedge \omega^2$, by Cartan's lemma we may write

$$(3) \quad \omega_i^r = \sum h_{ij}^r \omega^j, \quad h_{ij}^r = h_{ji}^r, \quad i, j = 1, 2.$$

From these we obtain

$$(4) \quad d\omega^i = -\sum \omega_j^i \wedge \omega^j,$$

$$(5) \quad d\omega_2^1 = \left\{ c + \sum_r \det(h_{ij}^r) \right\} \omega^1 \wedge \omega^2,$$

$$(6) \quad d\omega_i^r = -\sum \omega_j^r \wedge \omega_i^j - \sum \omega_s^r \wedge \omega_i^s.$$

The second fundamental form h , the mean curvature vector H , the Gauss curvature K and the scalar normal curvature K_N are given by

$$(7) \quad h = \sum_{r,i,j} h_{ij}^r \omega^i \omega^j e_r,$$

$$(8) \quad H = \frac{1}{2} \sum (h_{11}^r + h_{22}^r) e_r,$$

$$(9) \quad K = c + \sum_r \det(h_{ij}^r),$$

$$(10) \quad K_N = \sum_{r,s=3}^m \left\{ \sum_{i=1}^2 (h_{1i}^r h_{2i}^s - h_{1i}^s h_{2i}^r) \right\}^2,$$

respectively. A surface M is *minimal* if $H=0$ identically on M .

3. Proof of the Main Theorem. We first prove the following lemmas:

LEMMA 1. *Let M be a minimal surface of $R^m(c)$. If there exists a unit normal vector field e which is parallel in the normal bundle, then $M = M_1 \cup M_2$, where M_1 is an open subset of M with $K_N = 0$ and $M_2 = \{p \in M : \langle h(X, Y), e \rangle = 0 \text{ for all tangent vectors } X, Y \text{ in } T_p(M)\}$.*

PROOF. Let $M_1 = \{p \in M : \exists X, Y \in T_p(M) \text{ such that } \langle h(X, Y), e \rangle \neq 0\}$. Then M_1 is an open subset of M . We choose our frame field in such a way that

$$(11) \quad e_3 = e, \quad h_{12}^3 = 0,$$

on M_1 . Then we have

$$(12) \quad h_{11}^3 = -h_{22}^3 \neq 0$$

everywhere on M_1 . Since e is parallel in the normal bundle, we have

$$(13) \quad \omega_r^3 = 0.$$

By taking exterior differentiation of (13) and applying (4), (6) and (13), we obtain

$$(14) \quad h_{12}^r h_{11}^3 = 0, \quad \text{for } r = 4, \dots, m.$$

From (10), (12) and (14), we see that the scalar normal curvature $K_N = 0$ identically on M_1 . This proves Lemma 1.

LEMMA 2. *If M is a minimal surface of $R^m(c)$ with constant Gauss curvature and zero scalar normal curvature, then M is either flat and $c \geq 0$ or totally geodesic.*

PROOF. Since the scalar normal curvature $K_N = 0$ identically on M , we have $\sum_{i=1}^2 (h_{1i}^r h_{2i}^s - h_{1i}^s h_{2i}^r) = 0$ for $r, s = 3, \dots, m$. Thus, we may choose our local frame field in such a way that

$$(15) \quad h_{12}^r = 0, \quad r = 3, \dots, m.$$

Therefore we obtain

$$(16) \quad \omega_1^r = h_{11}^r \omega^1, \quad \omega_2^r = -h_{11}^r \omega^2.$$

Taking exterior differentiation of the first equation of (16), we have

$$(17) \quad dh_{11}^r \wedge \omega^1 + 2h_{11}^r d\omega^1 = \sum h_{11}^s \omega^1 \wedge \omega_s^r.$$

Multiplying (17) by h_{11}^r and summing up on r , we obtain

$$\sum (h_{11}^r dh_{11}^r) \wedge \omega^1 + 2 \sum (h_{11}^r)^2 d\omega^1 = \sum (h_{11}^r h_{11}^s) \omega^1 \wedge \omega_s^r.$$

Since $\omega_s^r = -\omega_r^s$, this equation implies

$$(18) \quad \sum (h_{11}^r dh_{11}^r) \wedge \omega^1 + 2 \sum (h_{11}^r)^2 d\omega^1 = 0.$$

On the other hand, the constancy of the Gauss curvature implies

$$(19) \quad \sum h_{11}^r dh_{11}^r = 0.$$

Therefore, combining (18) and (19), we obtain

$$(20) \quad \sum (h_{11}^r)^2 d\omega^1 = 0.$$

Similarly, by taking exterior differentiation of the second equation of (16), we obtain

$$(21) \quad \sum (h_{11}^r)^2 d\omega^2 = 0.$$

Put $V = \{p \in M : d\omega^1 \neq 0 \text{ or } d\omega^2 \neq 0\}$. Then V is an open subset of M . If $V = \emptyset$, then $d\omega^1 = d\omega^2 = 0$ identically on M . Hence, by (4), (5) and (9), we see that $K=0$ on M . Now, suppose that $V \neq \emptyset$, and V_1 is a component of V . Then we have

$$(22) \quad h_{11}^r = 0, \quad r = 3, \dots, m,$$

on V_1 . This implies that V_1 is totally geodesic in $R^m(c)$. If $c=0$, then $K=0$ identically on M . Now assume that $c \neq 0$. Then V_1 is of constant curvature c . This implies that $\text{closure}(V) = M$. Thus M is totally geodesic. This proves Lemma 2.

LEMMA 3. *Let M be a surface in a 3-dimensional totally geodesic submanifold $R^3(c)$ of $R^m(c)$ with unit normal vector $\bar{e}$ in $R^3(c)$. If e is a unit normal vector field of M in $R^m(c)$ which is parallel in the normal bundle, then there exist a constant unit normal vector field a in $R^m(c)$ and a constant angle α such that $e = (\cos \alpha)\bar{e} + (\sin \alpha)a$.*

This lemma can be proved immediately.

Now, we return to the proof of the Main Theorem. Let M be a minimal surface of a space form $R^m(c)$ with constant Gauss curvature K . If there exists a nonconstant unit normal vector field e which is parallel in the normal bundle, then by Lemma 1 we see that $M = M_1 \cup M_2$, where M_1 is open with scalar normal curvature $K_N = 0$ and $M_2 = M - M_1 = \{p \in M : \langle h(X, Y), e \rangle = 0 \text{ for all tangent vectors } X, Y \text{ at } p\}$. Since e is nonconstant, M_1 is nonempty. For each $p \in M_1$, there exist tangent vectors X, Y at p such that $\langle h(X, Y), e \rangle \neq 0$. Hence by Lemma 2 we see that M is flat and not totally geodesic in $R^m(c)$. Therefore by (9) we obtain $c > 0$, i.e., $R^m(c)$ is an m -sphere $S^m(c)$. Now let U be a component of M . Then,

by $K=K_N=0$, U is an open piece of a Clifford torus in a great 3-sphere $S^3(c)$ [3]. Let $\bar{e}$ denote the unit normal vector field of U in $S^3(c)$ and e_1, e_2 be in the principal directions of $\bar{e}$. Then we can find that the second fundamental form in the direction of $\bar{e}$ is given by

$$(\langle h(e_i, e_j), \bar{e} \rangle) = \begin{pmatrix} c^{1/2} & 0 \\ 0 & -c^{1/2} \end{pmatrix}.$$

Therefore by Lemma 3 we see that the second fundamental form in the direction of e is given by

$$(\langle h(e_i, e_j), e \rangle) = \begin{pmatrix} c^{1/2} \cos \alpha & 0 \\ 0 & -c^{1/2} \cos \alpha \end{pmatrix},$$

where $\cos \alpha$ is a nonzero constant. Therefore, by the continuity of the second fundamental form h on M and the definition of M_2 , we see that $M_1=M$. This implies that M is an open piece of a Clifford torus in a 3-sphere $S^3(c) \subset S^m(c)$. This completes the proof of the theorem.

REFERENCES

1. E. Calabi, *Isometric imbedding of complex manifolds*, Ann. of Math. (2) **58** (1953), 1–23. MR **15**, 160.
2. M. P. do Carmo and N. R. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math. (2) **93** (1971), 43–62.
3. K. Kenmotsu, *Some remarks on minimal submanifolds*, Tôhoku Math. J. (2) **22** (1970), 240–248. MR **42** #3688.
4. H. B. Lawson, Jr., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. (2) **89** (1969), 187–197. MR **38** #6505.
5. R. Osserman and S. S. Chern, *Complete minimal surfaces in euclidean n -space*, J. Analyse Math. **19** (1967), 15–34. MR **37** #2103.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823