IRREDUCIBLE 3-MANIFOLDS WHOSE ORIENTABLE COVERS ARE NOT PRIME¹

W. H. ROW, JR.

ABSTRACT. J. L. Tollefson has asked if every closed covering space of a prime 3-manifold is prime. In the present paper, the author provides a negative answer by constructing infinitely many topologically distinct, irreducible, closed 3-manifolds with the property that none of their orientable covering spaces are prime. These 3-manifolds are distinguished by the maximum number of disjoint, nonparallel, 2-sided projective planes that they contain. The author does not know if every closed covering space of a prime, orientable 3-manifold is prime.

1. Preliminaries. We will work in the piecewise linear category. Manifolds will be connected but not necessarily compact, orientable, or without boundary. *Surfaces* are compact 2-manifolds. A manifold N is *properly embedded* in a manifold M if $N \subset M$ and $N \cap \text{Bd } M = \text{Bd } N$. An $(n-1)$ -manifold N , properly embedded in an n -manifold M , is *2-sided* if N separates some connected neighborhood of N in M .

A 3-manifold M is *irreducible* if every 2-sphere in M bounds a 3-cell in M . A 3-manifold M is *prime* if every separating 2-sphere in M bounds a 3-cell in M . A properly embedded surface S in a 3-manifold M is *incompressible* provided (1) if D is a 2-cell in M such that $D \cap S = \text{Bd } D$, then $\text{Bd } D$ bounds a 2-cell in S , and (2) if S is a 2-sphere, then S does not bound a 3-cell in M . The unit interval $[0, 1]$ will be denoted by I .

If $S_1, \dots, S_m$ are disjoint, closed surfaces in a 3-manifold M , we say N is a *parallelity component* of $M - (S_1 \cup \dots \cup S_m)$ if for some closed surface S and some $S_i, S_j, i \neq j$, the closure of N is homeomorphic to $S \times I$ and S_i, S_j correspond to $S \times \{0\}, S \times \{1\}$, respectively. If $M - (S_1 \cup S_2)$ has a parallelity component, then we say S_1 and S_2 are *parallel*. A consequence of Lemma 1 is that if S_1 and S_2 are parallel in M , then $S_1, \dots, S_m$ have a parallelity component in M .

Received by the editors April 15, 1971.

AMS 1970 subject classifications. Primary 55A10, 57A10; Secondary 57C99, 55A40.

Key words and phrases. 3-manifold, irreducible, prime, covering space, incompressible, 2-sided projective plane, cube-with-a-knotted-hole.

¹ This research was partially supported by grant NSF GP-19853.

© American Mathematical Society 1972

Lemma 1 is a special case of a more general result of Haken [1, pp. 91–96].

LEMMA 1. *If S is a closed 2-manifold and T is an incompressible surface in the interior of $S \times I$, then T is parallel to $S \times \{0\}$.*

For the following two lemmas let $T_1, \dots, T_l$ be disjoint, 2-sided, incompressible, properly embedded surfaces in a 3-manifold M . We will follow Waldhausen [6, p. 57] in using $U(\dots)$ for nice regular neighborhoods.

LEMMA 2. *M is irreducible if and only if the closure of each component of $M - \bigcup U(T_i)$ is irreducible.*

LEMMA 3. *If $P_1, \dots, P_m$ are disjoint, 2-sided projective planes in $\text{Int } M$ and M is irreducible, then there is a homeomorphism h of M onto itself, fixed on $\text{Bd } M$, such that $h(P_1), \dots, h(P_m)$ do not intersect $T_1, \dots, T_l$.*

If M is orientable and compact, Lemma 2 appears in [6, p. 59]. The same techniques prove Lemma 2. In Lemma 3, since P_i and T_j are 2-sided, any “general position” intersection curve J of P_i and T_j is 2-sided on P_i . Hence J bounds a 2-cell on P_i . Using this fact the techniques used for Lemma 2 also prove Lemma 3.

LEMMA 4. *Let N be a closed 3-manifold that contains a 2-sided projective plane. Suppose M is a prime, orientable cover of N . Then M is homeomorphic to $S^2 \times S^1$.*

PROOF. Let p be a covering map from M to N . Suppose P is a 2-sided projective plane in N . Each component of $p^{-1}(P)$ is a covering space for P embedded in M as a 2-sided subset and hence a 2-sphere. Let S be one of the components of $p^{-1}(P)$. If S separates M , then S bounds a 3-cell C in M . We may suppose $p^{-1}(P) \cap \text{Int } C$ is empty. Therefore $p(C)$ is a compact 3-manifold in N with $\text{Bd}(p(C)) = P$, an impossibility. Since M contains a nonseparating 2-sphere, Milnor [4, Lemma 1] has shown M is homeomorphic to $S^2 \times S^1$.

2. Irreducible 3-manifolds that contain 2-sided projective planes. We will construct a 3-manifold Q_k for each positive integer k . Subscripts will run from 1 to k throughout this section. Let $a_1, \dots, a_k$ denote distinct singleton subsets of the 1-sphere S^1 . Suppose J is a nonseparating simple closed curve in P^2 , real projective 2-space. Then $K_i = U(J \times a_i) \subset P^2 \times S^1$ is a solid Klein bottle. Let E_i be a properly embedded 2-cell in K_i that intersects $J \times a_i$ in exactly one piercing point. Then $C_i = U(E_i)$ in K_i and the closure of $K_i - C_i$ are both 3-cells. Note that $A_i = C_i \cap \text{Bd } K_i$ is an annulus. Then $N_k = P^2 \times S^1 - \bigcup \text{Int } K_i$ is a 3-manifold. Let B_i be a 3-cell

in $C_i - A_i$ such that $\text{Bd } B_i$ and $\text{Bd } C_i$ intersect in two 2-cells, one in each component of $\text{Bd } C_i - A_i$, and the closure of $C_i - B_i$ is a cube-with-a-knotted-hole L_i . Let $M_k = N_k \cup (\bigcup L_i)$. In the terminology of [3, pp. 379–380] we obtain M_k from N_k by attaching cubes-with-a-knotted-hole L_i to N_k along annuli A_i .

Note. N_k and M_k are irreducible, both being the closure of $P^2 \times S^1$ minus disjoint, solid Klein bottles. A_i is not contractible in M_k , since otherwise, using the Loop Theorem, we could find a nonseparating 2-sphere in $P^2 \times S^1$. Hence A_i is an incompressible, properly embedded surface in M_k .

Let $Q_k = 2M_k$, the *double* of M_k , obtained by sewing two copies of M_k (denoted by M_k and M'_k) together along their boundaries by the identity map. If X is a subset of M_k , we will denote the corresponding subset of M'_k by X' . We consider M_k and M'_k both to be subsets of Q_k .

Since the orientable double cover of Q_k is closed, Theorem 5 answers a question of J. L. Tollefson [5, p. 106] in the negative.

REMARK. Each Q_k contains 2-sided projective planes. It may be of interest to note that if the Poincaré conjecture is true, Tollefson's question has an affirmative answer for 3-manifolds that do not contain 2-sided projective planes.

THEOREM 5. $Q_1, \dots, Q_k, \dots$ are topologically distinct, irreducible, closed 3-manifolds such that no orientable covering space of Q_k is prime.

PROOF. First we will show Q_k is irreducible. In view of Lemma 2 and the Note, it is sufficient to show that each component of $\text{Bd } M_k$ is incompressible in Q_k . Suppose some component of $\text{Bd } M_k$ is compressible in Q_k . Then there is a 2-cell $D \subset Q_k$ such that $D \cap \text{Bd } M_k = \text{Bd } D$ and $\text{Bd } D$ does not bound a 2-cell in $\text{Bd } M_k$. (Note $\text{Bd } D$ is not necessarily contained in the component of $\text{Bd } M_k$ we assumed to be compressible.) We may assume that $D \subset M_k$. Returning to the notation we used in defining M_k , if $\text{Bd } D$ is contained in the component of $\text{Bd } M_k$ that intersects L_i , let A, A^* denote the annuli obtained by taking the closures of $\text{Bd } L_i - A_i$ and $\text{Bd } K_i - A_i$, respectively. Then $\text{Bd } D \subset A \cup A^*$.

$\text{Bd } D$ must intersect $\text{Bd } A_i = \text{Bd } A = \text{Bd } A^*$. If not, then $\text{Bd } D$ is contained in A or A^* . Since both A and A^* are not contractible in M_k , $\text{Bd } D$ bounds a 2-cell in $\text{Bd } M_k$.

After performing a small ambient isotopy we can assume D and A_i are in general position. The components of $D \cap A$ and $D \cap A^*$ are arcs. We will call a properly embedded arc in an annulus a *spanning* arc if the endpoints of the arc lie in different boundary components of the annulus. We can assume the components of $D \cap A$ and $D \cap A^*$ are spanning arcs, since nonspanning arcs can easily be removed by an ambient isotopy.

There exist an arc Z in $D \cap A_i$ and 2-cells F_1 and F_2 such that $F_1 \cup F_2 = D$, $F_1 \cap F_2 = Z$, and $A_i \cap \text{Bd } F_1 = Z$. Then Z is a spanning arc of A_i since the closure of $\text{Bd } F_1 - Z$ is a spanning arc of either A or A^* .

If $\text{Bd } F_1 \subset A_i \cup A$, then $\text{Bd } F_1$ is a nontrivial simple closed curve on $\text{Bd } L_i$. Since the inclusion map from L_i into M_k induces a monomorphism on fundamental groups, an application of the Loop Theorem shows $\text{Bd } F_1$ must bound a 2-cell in L_i . But this contradicts the fact L_i is a cube-with-a-knotted-hole.

If $\text{Bd } F_1 \subset A_i \cup A^*$, then $\text{Bd } F_1$ is a simple closed curve on $\text{Bd } K_i$ that bounds an annulus with $J \times a_i$ in K_i . But then $J \times a_i$ is contractible in $P^2 \times S^1$. Hence K_i is a solid torus, a contradiction. We have now shown that Q_k is irreducible.

Let $P(M)$ denote the maximum number of disjoint, 2-sided, non-parallel projective planes that can be embedded in a compact 3-manifold; M. Haken ([1], [2]) has shown $P(M)$ exists. However we do not need Haken's complete result for Lemma 6. $P(M)$ is clearly a topological property of M . Lemma 6 shows $Q_1, \dots, Q_k, \dots$ are distinct.

LEMMA 6. $P(Q_k) = 2k$.

PROOF. Let $S_1, \dots, S_l$ be disjoint, 2-sided projective planes in Q_k . Returning to the notation used in defining N_k , if K_i is properly chosen, $P^2 \times a_i$ intersects N_k in a 2-cell D_i . Then the closures $R_1, \dots, R_k$ of the k components of $N_k - \bigcup (U(D_i))$ are each homeomorphic to $P^2 \times I$. Note that the components of $\text{Bd } M_k$, the annuli A_i , the annuli A'_i , the 2-cells D_i , and the 2-cells D'_i are incompressible, 2-sided, properly embedded surfaces in Q_k, M_k, M'_k, N_k , and N'_k , respectively. By repeated applications of Lemma 3 we can find a homeomorphism h of Q_k onto itself such that $h(S_1), \dots, h(S_l)$ miss the union of $\text{Bd } M_k$ with the A_i 's, A'_i 's, D_i 's, and D'_i 's. Hence we may assume that $S_1, \dots, S_l$ are contained in the 3-manifolds $R_1, \dots, R_k, R'_1, \dots, R'_k$. The cubes-with-a-knotted-hole $L_1, \dots, L_k, L'_1, \dots, L'_k$ cannot contain 2-sided projective planes. Lemma 1 implies that if $l > 2k$, then some S_i and S_j are parallel. So $P(Q_k) \leq 2k$.

Let h_i be a homeomorphism from $P^2 \times I$ onto R_i . Let $P_i = h_i(P^2 \times \{\frac{1}{2}\})$. Then $P_1, \dots, P_k, P'_1, \dots, P'_k$ are disjoint, 2-sided projective planes in Q_k . Since $\bigcup P_i$ and $\bigcup P'_i$ each fail to separate Q_k , only a P_i and a P'_j could bound a parallelity component N . But $\text{Bd } M_k$ separates P_i and P'_j in Q_k . So N contains a component of $\text{Bd } M_k$, an incompressible surface of Euler characteristic zero. This contradicts Lemma 1. Hence $P(Q_k) \geq 2k$.

All that remains to prove Theorem 5 is to show no orientable covering space of Q_k is prime. By Lemma 4 the only orientable, prime cover Q_k could have is $S^2 \times S^1$. Let p be a covering map from $S^2 \times S^1$ to Q_k . Now $p^{-1}(\text{Bd } M_k)$ is a collection of incompressible, closed surfaces of Euler

characteristic zero. But the only incompressible surfaces $S^2 \times S^1$ contains are 2-spheres. Hence $S^2 \times S^1$ cannot be a cover of Q_k . The proof is complete.

REFERENCES

1. Wolfgang Haken, *Some results on surfaces in 3-manifolds*, Studies in Modern Topology, Math. Assoc. Amer., distributed by Prentice-Hall, Englewood Cliffs, N.J., 1968, pp. 39–98. MR 36 #7118.
2. ———, *Erratum for some results on surfaces in 3-manifolds* (mimeographed notes).
3. D. R. McMillan, Jr. and Harry Row, *Tangled embeddings of one-dimensional continua*, Proc. Amer. Math. Soc. **22** (1969), 378–385. MR 39 #7571.
4. J. W. Milnor, *A unique decomposition theorem for 3-manifolds*, Amer. J. Math. **84** (1962), 1–7. MR 25 #5518.
5. J. L. Tollefson, *On 3-manifolds that cover themselves*, Michigan Math. J. **16** (1969), 103–109. MR 39 #4853.
6. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR 36 #7146.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916