EXTENDING A COMPLETE CONVEX METRIC

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ABSTRACT. A metric D is convex if for every two points x, z there is a third point y such that $D(x, y) + D(y, z) = D(x, z)$. A generalized continuum is a connected, locally compact, separable metric space. It is shown that if M_1 is a space with a complete convex metric D_1 and M_2 is a locally connected generalized continuum whose intersection with M_1 is nonempty and compact, there is a complete convex metric for $M_1 \cup M_2$ that extends D_1 . Using this result, four classes of locally connected generalized continua are characterized by the type of complete convex metric they admit.

1. Introduction. A metric D is said to be *convex* if for every two points x, z there is a third point y , called a *between point* of x and z , such that $D(x, y) + D(y, z) = D(x, z)$. If in addition $D(x, y) = D(y, z) = D(x, z)/2$, the point y is called a *midpoint* of x and z . A *linear set* in a metric space is one that is isometric with a subset of the real line; a linear arc is called a *segment*. A *generalized continuum* is a connected, locally compact, separable metric space. A *Peano continuum* is a connected, locally connected, compact metric space.

In 1949 Bing [3] and Moise [9] independently solved a long-standing problem by showing that every Peano continuum admits a convex metric. Further, Bing has shown [2] that *if M_1 and M_2 are intersecting Peano continua and D_1 is a convex metric for M_1 , there is a convex metric D_3 for $M_1 \cup M_2$ that extends D_1* . It is understood that the topology induced by D_3 on $M_1 \cup M_2$ preserves the given topology of M_2 .

In 1955 Tominaga and Tanaka [10] showed, by a method similar to that which Bing used to prove the convexifiability of Peano continua, that every locally connected generalized continuum admits a complete convex metric. The present paper addresses the question: Under what

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conditions can a complete convex metric be extended to a locally connected generalized continuum? By defining the metric D_3 of Bing [2] in a more general setting and by noting a special property of this metric, the following result is established: *If M_1 is a space with a complete convex metric D_1 and M_2 is a locally connected generalized continuum whose intersection with M_1 is nonempty and compact, there is a complete convex metric D_3 for $M_1 \cup M_2$ that extends D_1 .* In this case, the topology induced by D_3 is the largest one on $M_1 \cup M_2$ that preserves the given topology of each M_i .

2. Defining the metric. Let M_1 and M_2 be intersecting Peano continua and let D_i be a convex metric for M_i . In [2], the metric D_3 was defined in the following way. Let $F(x)$, $x > 0$, be a real-valued function which is never less than

$$\sup\{D_1(p, q) : p, q \in M_1 \cap M_2, D_2(p, q) \leq x\}$$

and such that $F(x)$ approaches zero as x approaches zero, $F'(x)$ is a continuous monotone nonincreasing function with $F'(x) > 1$, and the improper integral

$$\int_0^d F'(x) dx$$

exists for each $d > 0$. For every two points x, y of M_2 let $A(x, y)$ be the set of all D_2 rectifiable arcs C from x to y that lie, except for possibly their endpoints, in $M_2 \setminus M_1$ and for which the (possibly improper) Riemann integral

$$\int_C F'[D_2(p(s), M_1)] ds$$

exists. Here s denotes D_2 length along C from a fixed endpoint and $p(s)$ is the point of C whose D_2 distance along C from the fixed endpoint is s . If x lies in $M_2 \setminus M_1$, y is a point of M_1 such that $D_2(x, y) = D_2(x, M_1)$, and $[xy]$ is a D_2 segment from x to y , then

$$\int_{[xy]} F'[D_2(p(s), M_1)] ds$$

exists and has the value $F[D_2(x, M_1)]$. For all points x, y of M_2 with $A(x, y) \neq \emptyset$, let

$$D_0(x, y) = \inf \left\{ \int_C F'[D_2(p(s), M_1)] ds : C \in A(x, y) \right\}.$$

If $D_0(x, y)$ exists for two points x and y then $D_0(x, y) \geq D_2(x, y)$, and $D_0(x, y) \geq D_1(x, y)$ if x and y are points of M_1 . Define D_3 on $M_1 \cup M_2$ as

follows: if $x, y \in M_1$, then $D_3(x, y) = D_1(x, y)$; if $x \in M_1$ and $y \in M_2 \setminus M_1$, then $D_3(x, y) = D_3(y, x) = \inf\{D_1(x, a) + D_0(a, y) : a \in M_1, A(a, y) \neq \emptyset\}$; if $x, y \in M_2 \setminus M_1$, then $D_3(x, y)$ is the minimum of $D_0(x, y)$ and

$$\inf\{D_0(x, a) + D_1(a, b) + D_0(b, y) : a, b \in M_1, A(x, a) \neq \emptyset, A(b, y) \neq \emptyset\};$$

if $x = y$, then $D_3(x, y) = 0$. It follows that D_3 is a metric on $M_1 \cup M_2$, whose restriction to M_2 is equivalent to D_2 . The topology induced on $M_1 \cup M_2$ by D_3 is such that $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets.

3. Properties of the metric. For the definitions and properties of §2 to hold, it is not necessary that M_1 and M_2 be compact. It suffices that $M_1 \cup M_2$ be nonempty and compact and that every two points of M_2 lie on a D_2 segment. Menger [8] has shown that such segments exist if D_2 is complete and convex. For Lemmas 1–4, let M_1 be any space with a complete convex metric D_1 , let M_2 be a locally connected generalized continuum with a complete convex metric D_2 such that $M_1 \cup M_2$ is nonempty and compact, and let D_0 and D_3 be as defined in §2.

LEMMA 1. *If x is in $M_2 \setminus M_1$ and y is a point of M_1 such that $D_2(x, y) = D_2(x, M_1)$, then $D_0(x, y) = \inf\{D_0(x, a) : a \in M_1, A(x, a) \neq \emptyset\}$.*

PROOF. With x and y as given, it suffices to show that for any point a in M_1 with $A(x, a) \neq \emptyset$ and for any $C \in A(x, a)$, then

$$(1) \quad \int_{[xy]} F'[D_2(q(s), M_1)] ds \leq \int_C F'[D_2(p(s), M_1)] ds,$$

where $[xy]$ is a D_2 segment from x to y , $q(s)$ is the point u of $[xy]$ with $D_2(y, u) = s$, and $p(s)$ is the point of C whose D_2 distance along C from a is s . In fact, inequality (1) shows that

$$D_0(x, y) = \int_{[xy]} F'[D_2(q(s), M_1)] ds,$$

obtained by setting $a = y$.

Let α and β be the D_2 lengths of $[xy]$ and C , respectively. Since

$$D_2(p(s), M_1) \leq D_2(p(s), a) \leq s$$

and F' is monotone nonincreasing, then

$$F'[D_2(q(s), M_1)] = F'(s) \leq F'[D_2(p(s), M_1)] \quad \text{for } 0 < s \leq \beta.$$

Further, since $0 < \alpha = D_2(x, M_1) \leq \beta$, then

$$\int_0^\alpha F'[D_2(q(s), M_1)] ds \leq \int_0^\beta F'[D_2(p(s), M_1)] ds$$

follows by elementary properties of improper integrals, and is the desired inequality (1).

LEMMA 2. *For every point x of M_2 , $D_3(x, M_1) \geq D_2(x, M_1)$.*

PROOF. Assume x is in $M_2 \setminus M_1$. Let y be a point of M_1 such that $D_2(x, y) = D_2(x, M_1)$ and let z be any point of M_1 . Then from Lemma 1 and §2,

$$\begin{aligned} D_3(x, z) &= \inf\{D_0(x, a) + D_1(a, z) : a \in M_1, A(x, a) \neq \emptyset\} \\ &\geq \inf\{D_0(x, a) : a \in M_1, A(x, a) \neq \emptyset\} = D_0(x, y) \\ &\geq D_2(x, y) = D_2(x, M_1). \end{aligned}$$

Since z is arbitrary, it follows that $D_3(x, M_1) \geq D_2(x, M_1)$.

LEMMA 3. *Every closed and D_3 bounded subset of M_2 is compact.*

PROOF. By Lemma 2 such a set is also D_2 bounded, hence is compact according to the generalized Bolzano-Weierstrass theorem proved by Lelek and Mycielski [6] for a locally compact space with a complete convex metric.

LEMMA 4. *D_3 is complete and convex.*

PROOF. The completeness of D_3 is a result of Lemma 3 and the completeness of D_1 . Convexity is proved by applying the local compactness of M_2 in much the same way that compactness was used in the result of Bing [2].

THEOREM 1. *If M_1 is a space with a complete convex metric D_1 and M_2 is a locally connected generalized continuum whose intersection with M_1 is nonempty and compact, there is a complete convex metric for $M_1 \cup M_2$ that extends D_1 .*

PROOF. Tominaga and Tanaka [10] have shown that M_2 has a complete convex metric D_2 . If D_3 is constructed from D_1 and D_2 as in §2, then D_3 extends D_1 , is equivalent to D_2 on M_2 , and is complete and convex by Lemma 4.

It should be noted that the topology induced on $M_1 \cup M_2$ by the metric D_3 given above is the largest topology that preserves the original topologies of the spaces M_i , since $M_1 \setminus M_2$ and $M_2 \setminus M_1$ are separated sets in this topology.

4. Characterizing locally connected generalized continua by their complete convex metrics. By use of Theorem 1 certain classes of locally connected generalized continua can be identified by metric properties.

The following two definitions are equivalent to those discussed by Lelek and Nitka [7]. A complete convex metric is said to be *strongly convex* (SC) if every two points have a unique midpoint. A complete convex metric is *without ramifications* (WR) if whenever y is a midpoint of x and z , and also of x and z' , then $z=z'$. A preliminary proposition is given, whose proof requires only a slight modification of a proof of Blumenthal [4, p. 41].

PROPOSITION. *If p and q are two points of a complete convex metric space and if L is a linear set consisting of p , q , and between points of p and q , there is a segment from p to q containing L .*

THEOREM 2. *A locally connected generalized continuum contains no simple closed curve if and only if every complete convex metric for it is strongly convex.*

PROOF. Suppose the space M contains no simple closed curve. Let D be a complete convex metric for M , let x and z be two points of M , and suppose y and y' are midpoints of x and z . By the above proposition there are segments S and S' joining x to z and containing y and y' respectively. If $y \neq y'$, then $S \cup S'$ contains a simple closed curve. Thus, $y = y'$ and D is SC.

Now suppose that M contains a simple closed curve C . A homeomorphism from the unit circle in the plane onto C will induce a complete convex metric D_1 on C that is not SC—for example, the metric induced on C from the arc length metric on the unit circle. By Theorem 1 there is an extension of D_1 to a complete convex metric D_3 for M , and D_3 is not SC.

THEOREM 3. *A locally connected generalized continuum contains no simple triod if and only if every complete convex metric for it is without ramifications.*

PROOF. Let D be a complete convex metric for the space M , and suppose D is not WR. Then there are four points x, y, z, z' of M such that y is a midpoint of x and z , and also of x and z' . If $[xy]$, $[yz]$, and $[yz']$ are segments joining y to each of the three other points, then $[xy] \cup [yz] \cup [yz']$ must contain a simple triod.

Conversely, suppose M contains a simple triod T . Since T is homeomorphic with a triod T' in the plane composed of three equal line segments meeting at a point, the arclength metric on T' will induce a complete convex metric D_1 on T that is not WR. By Theorem 1, the metric D_1 extends to a complete convex metric D_3 for M , and D_3 is not WR.

THEOREM 4. *A locally connected generalized continuum is homeomorphic to an interval of the real line if and only if every complete convex metric for it is both strongly convex and without ramifications.*

PROOF. If the space M is homeomorphic to an interval of the real line, then by Theorems 2 and 3 every complete convex metric for M is SC and WR.

On the other hand suppose every complete convex metric for M is both SC and WR. Let D be a particular complete convex metric for M , let x and z be two points of M , and let y denote the unique midpoint of x and z . Suppose y' is a point of M distinct from y such that $D(x, y') = D(y', z)$. Let $[xy]$, $[yz]$, and $[yy']$ be segments from y to x , z and y' respectively. If either of the points x or z lies on $[yy']$ then $[xy] \cup [yz] \cup [yy']$ would contain a simple closed curve. If neither x nor z lies on $[yy']$ then $[xy] \cup [yz] \cup [yy']$ would contain a simple triod. Since by Theorems 2 and 3 both alternatives are impossible, then y itself is the only point of M such that $D(x, y) = D(y, z)$. Since M is a connected metric space in which every two points have exactly one point equidistant from them, then according to Berard [1], M is homeomorphic to an interval of the real line.

THEOREM 5. *A locally connected generalized continuum is a simple closed curve if and only if every complete convex metric for it is without ramifications but is not strongly convex.*

PROOF. If the space M is a simple closed curve, then by Theorem 3 every convex metric for M is WR. However, since M is not contractible to a point, M admits no SC metric [5].

If every complete convex metric on M is WR but not SC, then by Theorem 2, M must contain a simple closed curve C and, by Theorem 3, every point of M must lie on C .

If only compact spaces are considered, Theorems 2 through 5 yield results similar to those obtained by Toranzos [11].

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