FINITELY GENERATED SUBMODULES OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. Suppose $[\mathcal{E}^m(\Omega)]^p$ is the Cartesian product of the space of real-valued m -times continuously differentiable functions on an open set Ω in R^n with itself p -times where m is finite and Ω is connected. $[\mathcal{E}^m(\Omega)]^p$ is a $\mathcal{E}^m(\Omega)$ -module. The finitely generated submodules of $[\mathcal{E}^m(\Omega)]^p$ are $\text{im}(F)$ where $F: [\mathcal{E}^m(\Omega)]^q \rightarrow [\mathcal{E}^m(\Omega)]^p$ is a $p \times q$ matrix $(f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$, $f_{ij} \in \mathcal{E}^m(\Omega)$. In the present paper, it is shown that $\text{im}(F)$ is closed in $[\mathcal{E}^m(\Omega)]^p$ if and only if the rank of the matrix $(f_{ij}(x))_{1 \leq i \leq p, 1 \leq j \leq q}$ is constant for $x \in \Omega$. Applications are made to systems of division problems for distributions.

1. Introduction. Let $\mathcal{E}^m(\Omega)$ denote the algebra of real-valued m -times continuously differentiable functions on an open set Ω in R^n equipped with the topology of uniform convergence of all derivatives of order $\leq m$ on all compact subsets of Ω . Throughout we assume that m is finite and Ω is connected.

Let $[\mathcal{E}^m(\Omega)]^p$ denote the Cartesian product of $\mathcal{E}^m(\Omega)$ with itself p -times equipped with the product topology. Then $[\mathcal{E}^m(\Omega)]^p$ is a Fréchet space and a $\mathcal{E}^m(\Omega)$ -module. We here characterize the finitely generated submodules of $[\mathcal{E}^m(\Omega)]^p$ which are closed in $[\mathcal{E}^m(\Omega)]^p$. Interest in closed finitely generated submodules of $[\mathcal{E}^m(\Omega)]^p$ stems primarily from their connection by duality with systems of division problems for distributions (see §3).

2. The main result. Consider the finitely generated submodule $M = \{g_1 \mathbf{f}_1 + \cdots + g_q \mathbf{f}_q : g_1, \dots, g_q \in \mathcal{E}^m(\Omega)\}$ of $[\mathcal{E}^m(\Omega)]^p$ where $\mathbf{f}_j = (f_{1j}, \dots, f_{pj}) \in [\mathcal{E}^m(\Omega)]^p$ for $1 \leq j \leq q$. Let F be the $p \times q$ matrix $(f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$. Then $F: [\mathcal{E}^m(\Omega)]^q \rightarrow [\mathcal{E}^m(\Omega)]^p$ and $\text{im}(F) = M$. Our main result is that $M = \text{im}(F)$ is closed in $[\mathcal{E}^m(\Omega)]^p$ if and only if the rank of $F(x) = (f_{ij}(x))_{1 \leq i \leq p, 1 \leq j \leq q}$ is constant for $x \in \Omega$.

The sufficiency of the condition that $\text{rank}(F(x))$ be constant is a direct

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consequence of Malgrange's characterization of the closure of a submodule of differentiable functions [1, Corollary 1.6, p. 25]. For continuous functions, that is, when $m=0$, the necessity of the condition that $\text{rank}(F(x))$ be constant can be shown using the open mapping theorem (see [4, Theorem 3.1]). However, in general, a different method of proof is required. Assuming that $M=\text{im}(F)$ is closed in $[\mathcal{E}^m(\Omega)]^p$, we have $\text{im}(F')=[\ker(F)]^\perp$ where $F': [\mathcal{E}'^m(\Omega)]^p \rightarrow [\mathcal{E}'^m(\Omega)]^q$ is the transpose of F . Assuming that $\text{rank}(F(x))$ is not constant on Ω , we find a q -tuple of distributions

$$(T_1, \dots, T_q) \in [\ker(F)]^\perp$$

such that for some j , $1 \leq j \leq q$, the support of T_j consists of a single point where the rank of F is "changing" and the order of T_j is "large". Therefore $F'(S_1, \dots, S_p) = (T_1, \dots, T_q)$ for some $(S_1, \dots, S_p) \in [\mathcal{E}'^m(\Omega)]^p$ which is impossible because the order of T_j is too large.

We first prove a lemma establishing the existence of a $(T_1, \dots, T_q) \in [\ker(F)]^\perp$ with the appropriate properties.

LEMMA 2.1. Suppose $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$, $f_{ij} \in \mathcal{E}^m(\Omega)$. Suppose

$$a \in \text{bd}\{x \in \Omega: \text{rank}(F(x))=r\} \quad \text{where } r = \max\{\text{rank}(F(x)): x \in \Omega\}.$$

Let $s = \text{rank}(F(a))$. Then there exists $(T_1, \dots, T_q) \in [\ker(F)]^\perp$ such that:

(1) for some $j_1, \dots, j_s$, $T_{j_1} = \dots = T_{j_s} = 0$ and the $j_1, \dots, j_s$ columns of $F(a)$ are linearly independent;

(2) for some j_{s+1} , $\text{supp}(T_{j_{s+1}}) = \{a\}$ and $\text{order}(T_{j_{s+1}}) = m - k$ where k is the largest integer $\leq m$ such that the determinant of every $(s+1) \times (s+1)$ submatrix of F with rank s at a vanishes at a together with all derivatives of order $\leq k$.

PROOF. If $s=0$, the lemma asserts that there exists $(T_1, \dots, T_q) \in [\ker(F)]^\perp$ such that for some j , $1 \leq j \leq q$, $\text{supp}(T_j) = \{a\}$ and $\text{order}(T_j) = m - k$ where k is the largest integer $\leq m$ such that every f_{ij} vanishes at a together with all derivatives of order $\leq k$. Since the proof for $s=0$ is similar to the proof for $s>0$, we assume $s>0$.

Let k be as in (2). If $k=m$, then there exists an $(s+1) \times (s+1)$ submatrix of F with rank s at a which is nonsingular at every point of a sequence $\{x_i\}$ in Ω with $x_i \rightarrow a$ since $a \in \text{bd}\{x \in \Omega: \text{rank}(F(x))=r\}$. If $k < m$, then there exists an $(s+1) \times (s+1)$ submatrix of F with rank s at a whose determinant has a zero of order exactly k at a by our choice of k . In each case, for simplicity suppose that $(f_{ij})_{1 \leq i, j \leq s+1}$ is the $(s+1) \times (s+1)$ submatrix of F and the first s columns of $(f_{ij}(a))_{1 \leq i, j \leq s+1}$ are linearly independent.

Consider $(g_1, \dots, g_q) \in \ker(F)$. Then

$$f_{11}g_1 + \dots + f_{1q}g_q = 0,$$

$$\begin{array}{cccc} \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \end{array}$$

$$f_{(s+1)1}g_1 + \dots + f_{(s+1)q}g_q = 0.$$

For $1 \leq i \leq s+1$, multiply the i th row of this system of equations by the cofactor of $f_{i(s+1)}$ in $(f_{ij})_{1 \leq i, j \leq s+1}$. Adding the equations thus obtained, we find that $d_1g_1 + \dots + d_qg_q = 0$ where for $1 \leq j \leq q$,

$$d_j = \det \begin{pmatrix} f_{11} & \dots & f_{1s} & f_{1j} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ f_{(s+1)1} & \dots & f_{(s+1)s} & f_{(s+1)j} \end{pmatrix}.$$

Clearly $d_1 = \dots = d_s = 0$ so $d_{s+1}g_{s+1} + \dots + d_qg_q = 0$ for all $(g_1, \dots, g_q) \in \ker(F)$.

Define $D: [\mathcal{E}^m(\Omega)]^{q-s} \rightarrow \mathcal{E}^m(\Omega)$ by $D(h_1, \dots, h_{q-s}) = d_{s+1}h_1 + \dots + d_qh_{q-s}$. Then whenever $(g_1, \dots, g_q) \in \ker(F)$, we have $(g_{s+1}, \dots, g_q) \in \ker(D)$. If $k=m$, then d_{s+1} is nonzero at every point of a sequence $\{x_i\}$ in Ω with $x_i \rightarrow a$, while if $k < m$, then each d_j , $s+1 \leq j \leq q$, has a zero of order at least k at a and d_{s+1} has a zero of order exactly k at a . Applying Lemma 3.1 of [2, p. 215] if $k=m$ and Lemma 3.2 of [2, p. 215] if $k < m$, we find that there exists $(T_1, \dots, T_{q-s}) \in [\ker(D)]^\perp$ such that for some j , $1 \leq j \leq q-s$, $\text{supp}(T_j) = \{a\}$ and $\text{order}(T_j) = m-k$. Then $(0, \dots, 0, T_1, \dots, T_{q-s}) \in [\ker(F)]^\perp$ and satisfies (1) and (2).

THEOREM 2.1. Suppose $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$, $f_{ij} \in \mathcal{E}^m(\Omega)$. The finitely generated submodule $M = \text{im}(F)$ of $[\mathcal{E}^m(\Omega)]^p$ is closed in $[\mathcal{E}^m(\Omega)]^p$ if and only if the rank of $F(x) = (f_{ij}(x))_{1 \leq i \leq p, 1 \leq j \leq q}$ is constant for $x \in \Omega$.

PROOF. Suppose $M = \text{im}(F)$ is closed in $[\mathcal{E}^m(\Omega)]^p$ but $\text{rank}(F(x))$ is not constant on Ω . Let $r = \max\{\text{rank}(F(x)) : x \in \Omega\}$. Since Ω is connected, there exists $a = (a_1, \dots, a_n) \in \text{bd}\{x \in \Omega : \text{rank}(F(x)) = r\}$. Let $s = \text{rank}(F(a))$. By Lemma 2.1, there exists $(T_1, \dots, T_q) \in [\ker(F)]^\perp$ satisfying (1) and (2) where we assume for the sake of simplicity that $j_1 = 1, \dots, j_s = s, j_{s+1} = s+1$.

Since $\text{im}(F)$ is closed in $[\mathcal{E}^m(\Omega)]^p$, the closed range theorem for Fréchet spaces implies that $\text{im}(F') = [\ker(F)]^\perp$ where $F': [\mathcal{E}'^m(\Omega)]^p \rightarrow [\mathcal{E}'^m(\Omega)]^q$ is the transpose of F . Hence there exists $(S_1, \dots, S_p) \in [\mathcal{E}'^m(\Omega)]^p$ with

$F'(S_1, \dots, S_p) = (T_1, \dots, T_q)$. Letting $f'_{ij} = f_{ji}$ for $1 \leq i \leq q$, $1 \leq j \leq p$, we have

$$f'_{11}S_1 + \dots + f'_{1p}S_p = T_1 = 0,$$

$$\begin{array}{cccc} \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \end{array}$$

$$f'_{s1}S_1 + \dots + f'_{sp}S_p = T_s = 0,$$

$$f'_{(s+1)1}S_1 + \dots + f'_{(s+1)p}S_p = T_{s+1}.$$

By (1), the columns of $(f_{ij}(a))_{1 \leq i \leq p, 1 \leq j \leq s}$ are linearly independent so s rows of $(f_{ij}(a))_{1 \leq i \leq p, 1 \leq j \leq s}$ are linearly independent, say for simplicity the first s rows. Then $\det(f'_{ij}(a))_{1 \leq i, j \leq s} = \det(f_{ij}(a))_{1 \leq i, j \leq s} \neq 0$. For $1 \leq i \leq s+1$, multiply the i th row in the above system of equations by the cofactor of $f'_{i(s+1)}$ in $(f'_{ij})_{1 \leq i, j \leq s+1}$. Adding the equations thus obtained we find that

$$g_1S_1 + \dots + g_pS_p = \det(f'_{ij})_{1 \leq i, j \leq s} T_{s+1},$$

where for $1 \leq j \leq p$,

$$g_j = \det \begin{pmatrix} f'_{11} & \dots & f'_{1s} & f'_{1j} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ f'_{(s+1)1} & \dots & f'_{(s+1)s} & f'_{(s+1)j} \end{pmatrix}.$$

By our choice of k in (2), the functions $g_1, \dots, g_p$ each vanish at a together with all derivatives of order $\leq k$. And by (2), $\text{supp}(T) = \{a\}$ and $\text{order}(T) = m - k$ where $T = \det(f'_{ij})_{1 \leq i, j \leq s} T_{s+1}$ since $\det(f'_{ij}(a))_{1 \leq i, j \leq s} \neq 0$.

The equation $g_1S_1 + \dots + g_pS_p = T$ together with Malgrange's characterization of the closure of a submodule of differentiable functions leads to a contradiction. To see this, define $G: \mathcal{E}^m(\Omega) \rightarrow [\mathcal{E}^m(\Omega)]^p$ by $G(h) = (g_1h, \dots, g_ph)$ and let $G': [\mathcal{E}'^m(\Omega)]^p \rightarrow \mathcal{E}'^m(\Omega)$ be the transpose of G . Then $G'(S_1, \dots, S_p) = g_1S_1 + \dots + g_pS_p = T$. Define $f \in \mathcal{E}^m(\Omega)$ by $f(x_1, \dots, x_n) = (x_1 - a_1)^l + \dots + (x_n - a_n)^l$ where l is an even integer $> m$. Define $g \in \mathcal{E}^m(\Omega)$ by $g(x_1, \dots, x_n) = (x_1 - a_1)^{\alpha_1} \dots (x_n - a_n)^{\alpha_n}$ where $\alpha_1, \dots, \alpha_n$ are chosen so that $\alpha_1 + \dots + \alpha_n = m - k$ and $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \delta_a$ appears with a nonzero coefficient in T .

Then $G'(fS_1, \dots, fS_p) = fT = 0$ and thus $(fS_1, \dots, fS_p) \in \ker(G') = N^\perp$ where $N = \text{im}(G)$ is a submodule of $[\mathcal{E}^m(\Omega)]^p$. Hence $(S_1, \dots, S_p) \in (fN)^\perp$. For $x \in \Omega$, let T_x^m be the natural mapping of $[\mathcal{E}^m(\Omega)]^p$ onto $[\mathcal{E}^m(\Omega)]^p/[J_x^m]^p$ where J_x^m is the ideal in $\mathcal{E}^m(\Omega)$ consisting of all functions which vanish at x together with all derivatives of order $\leq m$. Then $T_a^m(fN)$ and $T_a^m(gN)$ are both the zero submodule in $[\mathcal{E}^m(\Omega)]^p/[J_a^m]^p$. And for $x \in \Omega$, $x \neq a$, we have

$T_x^m(gN) \subset T_x^m(fN)$ since $f(x) \neq 0$. By Malgrange's theorem [1, Corollary 1.6, p. 25], we conclude that $\text{cl}(gN) \subset \text{cl}(fN)$ and hence $(fN)^\perp \subset (gN)^\perp$. Therefore $(S_1, \dots, S_p) \in (gN)^\perp$ and thus $(gS_1, \dots, gS_p) \in N^\perp = \ker(G')$. Hence $0 = G'(gS_1, \dots, gS_p) = gT$. But $gT \neq 0$ since $\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \delta_a$ appears with a nonzero coefficient in T . This contradiction completes the proof of the necessity that $\text{rank}(F(x))$ be constant on Ω .

Conversely, suppose $\text{rank}(F(x)) = r$ for all $x \in \Omega$. Consider $(h_1, \dots, h_p) \in \text{cl}(M)$. Then Malgrange's theorem says that $(h_1, \dots, h_p)$ is pointwise in M , that is, $T_x^m(h_1, \dots, h_p) \in T_x^m(M)$ for all $x \in \Omega$. This fact enables us to find $(g_1, \dots, g_q) \in [\mathcal{E}^m(\Omega)]^q$ with $F(g_1, \dots, g_q) = (h_1, \dots, h_p)$ and thus $(h_1, \dots, h_p) \in \text{im}(F) = M$.

Since the case $r=0$ is trivial, we assume $r>0$. There exists a locally finite open covering $\{\Omega_\alpha\}$ of Ω with the property that for each Ω_α there exists an $r \times r$ submatrix of F which is nonsingular at every $x \in \Omega_\alpha$. Using a partition of unity argument, it is easy to see that it suffices to find $(g_1, \dots, g_q) \in [\mathcal{E}^m(\Omega_\alpha)]^p$ with $F(g_1, \dots, g_q) = (h_1, \dots, h_p)$ in Ω_α for each α .

Suppose Ω_α is a member of the open covering. Then some $r \times r$ submatrix of F is nonsingular at every $x \in \Omega_\alpha$, say for simplicity $(f_{ij})_{1 \leq i, j \leq r}$. Then the system of equations

$$\begin{aligned} f_{11}g_1 + \dots + f_{1r}g_r &= h_1, \\ &\vdots \\ f_{r1}g_1 + \dots + f_{rr}g_r &= h_r, \end{aligned}$$

can be uniquely solved in Ω_α for functions $g_1, \dots, g_r \in \mathcal{E}^m(\Omega_\alpha)$. Consider $(g_1, \dots, g_q) \in [\mathcal{E}^m(\Omega_\alpha)]^q$ where $g_j = 0$ for $r < j \leq q$.

We claim that $F(g_1, \dots, g_q) = (h_1, \dots, h_p)$ in Ω_α , that is, $f_{i1}g_1 + \dots + f_{iq}g_q = h_i$ in Ω_α for $1 \leq i \leq p$. Since in Ω_α for $1 \leq i \leq p$ we have

$$h_i - \sum_{j=1}^q f_{ij}g_j = \det \begin{pmatrix} f_{11} & \dots & f_{1r} & h_1 \\ \vdots & & \vdots & \vdots \\ f_{r1} & \dots & f_{rr} & h_r \\ f_{i1} & \dots & f_{ir} & h_i \end{pmatrix} / \det(f_{ij})_{1 \leq i, j \leq r},$$

it suffices to prove that the determinant in the numerator is zero in Ω_α for $1 \leq i \leq p$.

Consider $x \in \Omega_a$. Since $(h_1, \dots, h_p)$ is pointwise in $M = \text{im}(F)$, the column vector $(h_i(x))_{1 \leq i \leq p}$ is linearly dependent on the columns of the matrix $(f_{ij}(x))_{1 \leq i \leq p, 1 \leq j \leq q}$. Since $(f_{ij}(x))_{1 \leq i, j \leq r}$ is nonsingular and $\text{rank}(F(x)) = r$, the last $q - r$ columns of $(f_{ij}(x))_{1 \leq i \leq p, 1 \leq j \leq q}$ are linearly dependent on the first r columns. Hence the column vector $(h_i(x))_{1 \leq i \leq p}$ is linearly dependent on the columns of the matrix $(f_{ij}(x))_{1 \leq i \leq p, 1 \leq j \leq r}$. Therefore

$$\det \begin{pmatrix} f_{11}(x) & \cdots & f_{1r}(x) & h_1(x) \\ \vdots & & \vdots & \vdots \\ f_{r1}(x) & \cdots & f_{rr}(x) & h_r(x) \\ f_{i1}(x) & \cdots & f_{ir}(x) & h_i(x) \end{pmatrix} = 0$$

for $1 \leq i \leq p$ which completes the proof of Theorem 2.1.

COROLLARY 2.1. Suppose $f_1, \dots, f_q \in \mathcal{E}^m(\Omega)$. The ideal I generated by $f_1, \dots, f_q$ in $\mathcal{E}^m(\Omega)$ is closed in $\mathcal{E}^m(\Omega)$ if and only if $f_1^2 + \dots + f_q^2$ never vanishes or $f_1^2 + \dots + f_q^2$ is identically zero.

PROOF. Let $F = (f_j)_{1 \leq j \leq q}$. Then $\text{rank}(F(x)) = 0$ for all $x \in \Omega$ if and only if $f_1^2 + \dots + f_q^2$ is identically zero in Ω . And $\text{rank}(F(x)) = 1$ for all $x \in \Omega$ if and only if $f_1^2 + \dots + f_q^2$ is never zero in Ω .

3. Applications. For $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$, $f_{ij} \in \mathcal{E}^m(\Omega)$, define $F: [\mathcal{D}'^m(\Omega)]^q \rightarrow [\mathcal{D}'^m(\Omega)]^p$ by

$$F(S_1, \dots, S_q) = \left(\sum_{j=1}^q f_{1j} S_j, \dots, \sum_{j=1}^q f_{pj} S_j \right).$$

Let $F_c: [\mathcal{E}'^m(\Omega)]^q \rightarrow [\mathcal{E}'^m(\Omega)]^p$ be the restriction of F to $[\mathcal{E}'^m(\Omega)]^q$. Let $F': [\mathcal{D}'^m(\Omega)]^p \rightarrow [\mathcal{D}'^m(\Omega)]^q$ (respectively $F'_c: [\mathcal{E}'^m(\Omega)]^p \rightarrow [\mathcal{E}'^m(\Omega)]^q$) be the transpose of F (respectively F_c).

Consider $F(S_1, \dots, S_q) = (T_1, \dots, T_p)$, that is, the system of division problems $f_{i1}S_1 + \dots + f_{iq}S_q = T_i$ for $1 \leq i \leq p$, where $(T_1, \dots, T_p) \in [\mathcal{D}'^m(\Omega)]^p$. In order that there exist a solution $(S_1, \dots, S_q) \in [\mathcal{D}'^m(\Omega)]^q$, it is necessary that $(T_1, \dots, T_p) \in [\ker(F')]^\perp$ since $\text{im}(F) \subset [\ker(F')]^\perp$. Equivalently, it is necessary that every relation between the rows of $(f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ be a relation between $(T_1, \dots, T_p)$, that is, if $\psi_1 \mathbf{f}_1 + \dots + \psi_p \mathbf{f}_p = (0, \dots, 0)$ where $\mathbf{f}_i = (f_{i1}, \dots, f_{iq})$ and $\psi_i \in \mathcal{D}'^m(\Omega)$ for $1 \leq i \leq p$, then $\psi_1 T_1 + \dots + \psi_p T_p = 0$. We are interested in those cases where for every $(T_1, \dots, T_p) \in [\ker(F')]^\perp$ there exists $(S_1, \dots, S_q) \in [\mathcal{D}'^m(\Omega)]^q$ satisfying $F(S_1, \dots, S_q) = (T_1, \dots, T_p)$ and thus are led to the following definition.

DEFINITION. $[\mathcal{D}'^m(\Omega)]^p$ is divisible by $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ if for every $(T_1, \dots, T_p) \in [\ker(F')]^\perp$ there exists $(S_1, \dots, S_q) \in [\mathcal{D}'^m(\Omega)]^q$ with $F(S_1, \dots, S_q) = (T_1, \dots, T_p)$.

Then $[\mathcal{D}'^m(\Omega)]^p$ is divisible by $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ if and only if $\text{im}(F) = [\ker(F')]^\perp$. A simple partition of unity argument establishes that $\text{im}(F) = [\ker(F')]^\perp$ if and only if $\text{im}(F_c) = [\ker(F'_c)]^\perp$. The closed range theorem for Fréchet spaces now provides the link between systems of division problems for distributions and finitely generated submodules of differentiable functions. For it says that $\text{im}(F_c) = [\ker(F'_c)]^\perp$ if and only if $\text{im}(F'_c)$, which is the submodule generated by $\mathbf{f}_1, \dots, \mathbf{f}_p$ in $[\mathcal{E}^m(\Omega)]^q$ where $\mathbf{f}_i = (f_{i1}, \dots, f_{iq})$ for $1 \leq i \leq p$, is closed in $[\mathcal{E}^m(\Omega)]^q$. Therefore Theorem 2.1 gives the following result.

THEOREM 3.1. $[\mathcal{D}'^m(\Omega)]^p$ is divisible by $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ if and only if $\text{rank}(F(x))$ is constant for $x \in \Omega$.

COROLLARY 3.1. $\text{im}(F) = [\mathcal{D}'^m(\Omega)]^p$ where $F = (f_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$ if and only if $\text{rank}(F(x)) = p$ for all $x \in \Omega$.

PROOF. $\text{im}(F) = [\mathcal{D}'^m(\Omega)]^p$ if and only if $[\mathcal{D}'^m(\Omega)]^p$ is divisible by F and $F': [\mathcal{D}'^m(\Omega)]^p \rightarrow [\mathcal{D}'^m(\Omega)]^q$ is one-to-one. It can be shown that $F': [\mathcal{D}'^m(\Omega)]^p \rightarrow [\mathcal{D}'^m(\Omega)]^q$ is one-to-one if and only if the set of $x \in \Omega$ for which $\text{rank}(F(x)) = p$ is dense in Ω (see [3, Lemma 3.1]). The corollary now follows immediately from Theorem 3.1.

BIBLIOGRAPHY

1. B. Malgrange, *Ideals of differentiable functions*, Tata Inst. Fund. Res. Studies in Math., no. 3, Tata Inst. of Fundamental Research, Bombay; Oxford Univ. Press, London, 1966. MR 35 #3446.
2. B. Roth, *Finitely generated ideals of differentiable functions*, Trans. Amer. Math. Soc. **150** (1970), 213–225. MR 41 # 7415.
3. ———, *Systems of division problems for distributions*, Trans. Amer. Math. Soc. **155** (1971), 493–504.
4. ———, *Submodules of $C(X) \times \dots \times C(X)$* , Proc. Amer. Math. Soc. **32** (1972), 543–548.

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