A CHARACTERIZATION OF SPACES ON WHICH ALL PATH MAPS ARE CONTINUOUS

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ABSTRACT. Here it is shown that all path maps on a topological space X are continuous if and only if X is a quotient image of a locally path connected metric space. This characterization strengthens results of E. Bedford [1].

1. Introduction. A function $f: X \rightarrow Y$ between the topological spaces X and Y is a *path map* if, whenever $\sigma: I \rightarrow X$ ($I = [0, 1]$) is a path in X , so is $f\sigma: I \rightarrow Y$ a path in Y . E. Bedford studied path maps in [1] and established the following theorem.

THEOREM 1.1. *If X is Hausdorff and first countable then every path map on X is continuous if and only if X is locally path connected.*

Using different methods we have been able to give a characterization (reminiscent of S. P. Franklin's characterization of sequential spaces [3]; see also [4]) of spaces on which all path maps are continuous as quotient images of locally path connected metric spaces. Thus we have proved:

THEOREM 1.2. *All path maps on a topological space X are continuous if and only if X is a quotient image of a locally path connected metric space.*

The extent to which this theorem strengthens Bedford's result is illustrated by the following corollary.

COROLLARY 1.3. *If all path maps on a topological space X are continuous, then X is locally path connected.*

This corollary removes the Hausdorff and first countable hypotheses from one half of Theorem 1.1. A slight modification of Bedford's proof (put $j_n = n$ if it does not exist as defined) will remove the Hausdorff assumption from the converse, but without first countability the converse is false.

EXAMPLE 1.4. Let X denote the space of all continuous functions $[0, 1] \rightarrow [0, 1]$ with the point-open topology.

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It is clear that X is Hausdorff and locally path connected. As an example of a discontinuous path map on X we offer the ordinary Riemann integral J . By dominated convergence, J is sequentially continuous. Thus if $\sigma: I \rightarrow X$ is a path, $J\sigma$ is sequentially continuous, thus a path.

2. Proofs. Recall that X is *locally path connected* if it has a basis of path connected open sets.

PROOF OF THEOREM 1.2. Let all path maps on X be continuous. Denote by X^I the set of all paths $I \rightarrow X$ and by Z the product set $X^I \times I$. Define a metric d on Z by

$$\begin{aligned} d((\sigma, t), (\sigma', t')) &= 1, & \text{if } \sigma \neq \sigma', \\ &= |t - t'|, & \text{if } \sigma = \sigma'. \end{aligned}$$

With this metric Z becomes the topological sum of the spaces $\{\sigma\} \times I$, where $\sigma: I \rightarrow X$ is a path. Consequently, Z is locally path connected, and the (onto) function $f: Z \rightarrow X$ defined by $f(\sigma, t) = \sigma(t)$ is continuous. Hence, the proof will be completed if f is shown to be a quotient. To this end let $g: X \rightarrow Y$ be any function for which the composition gf is continuous. Then, by construction, g is easily seen to be a path map on X , thus continuous. Therefore, by [2, p. 123], f is a quotient map.

Conversely, suppose that Z is a locally path connected metric space and $f: Z \rightarrow X$ a quotient map. If $g: X \rightarrow Y$ is a path map on X , then $gf: Z \rightarrow Y$ is a path map on Z , thus continuous by Theorem 1.1. But since f is a quotient map, g must also be continuous.

PROOF OF COROLLARY 1.3. If all path maps on X are continuous, represent X as a quotient image of a locally path connected metric space Z . The result now follows from the preservation of local path connectedness by quotient maps.

REFERENCES

1. E. Bedford, *Path maps and continuity*, Proc. Amer. Math. Soc. **24** (1970), 112–115. MR 40 #6517.
2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR 33 #1824.
3. S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115. MR 31 #5184.
4. P. W. Harley III, *On sequential spaces and maps* (submitted).

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