EQUIVALENT CONDITIONS FOR THE L_p CONVOLUTION THEOREM ON SEMISIMPLE GROUPS

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ABSTRACT. For certain semisimple Lie groups G , it is known that convolution by an $L_p(G)$ function, $1 \leq p < 2$, is a bounded operator on $L_2(G)$. This result is a consequence of the so-called "analytic continuation of the principal series" which has been carried out on these groups. However, this continuation procedure does not generalize readily to *arbitrary* semisimple groups. In an attempt to bypass the continuation and obtain the convolution theorem in an alternate manner, we derive in this paper several equivalent conditions for this L_p convolution theorem.

1. Introduction. In recent papers ([4a], [4b]) we derived, as consequences of Kunze and Stein's analytic continuation of the principal series for $SL(n, \mathbb{C})$ [3b], several applications to harmonic analysis on these groups. Among them are two results: (i) the L_p convolution theorem—the map $(f, h) \rightarrow f * h$ is a bounded bilinear mapping of $L_2 \times L_p$ into L_2 , $1 \leq p < 2$; and (ii) an analog of the Riemann-Lebesgue Lemma for L_p functions, $1 \leq p < 2$. In this paper, we shall show that these and several other conditions are actually equivalent propositions on real semisimple Lie groups.

2. Fourier analysis on semisimple groups. Let G be a connected semisimple Lie group with finite center, and let K be a maximal compact subgroup. Denote by $H_1, \dots, H_r$ a complete set of Cartan subgroups of G (the H_i are pairwise nonconjugate, and every Cartan subgroup is conjugate to one of them). Fix $H =$ a particular H_i for the moment, and set $\mathfrak{h} =$ its Lie algebra. We may assume that H is invariant under the Cartan involution of G determined by K . Then $H = BA$, where $B = H \cap K$ and $A = \exp \mathfrak{a}$, $\mathfrak{a} =$ the subalgebra of $\mathfrak{h}$ on which every root of $(\mathfrak{g}, \mathfrak{h})$ takes real values ($\mathfrak{g}$ of course is the Lie algebra of G). Let P be a cuspidal parabolic subgroup of G having A as its split component. Then P has a Langland's

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decomposition $P=MAN$, where M is a (disconnected) reductive Lie group and B is a compact Cartan subgroup of M . (The basic reference for the preceding is [2].)

The character group $\hat{A}$ of A may be identified with $\mathfrak{a}_R^*$ by $\nu(a) = \exp i\nu(\log a)$, $a \in A$, $\nu \in \mathfrak{a}_R^*$. Let $\hat{A}$ have the Euclidean topology determined by this identification. Next, let $\hat{M}_s$ = the unitary equivalence classes of square-integrable irreducible unitary representations of M . The structure of $\hat{M}_s$ was computed explicitly in a recent paper [4f]. Let $\hat{M}_s$ have the discrete topology. (By extending the results of [4e], it is possible to show that this agrees with the relative topology induced by the hull-kernel topology on all of $\hat{M}$.) Set $\Lambda = \hat{M}_s \times \hat{A}$ with the product topology. Note that if H is compact, then $P=G$, A is trivial and $\Lambda = \hat{G}_s$ is a discrete topological space.

Corresponding to each Cartan subgroup H_i , we obtain a topological space Λ_i in the manner just described. Let $\Lambda = \bigcup_{i=1}^r \Lambda_i$, a disjoint union. We make Λ into a Hausdorff topological space by declaring each Λ_i to be open. We denote by $d\lambda$ the canonical Borel measure on Λ ; i.e. on each Λ_i , $d\lambda$ is the product of the discrete measure on $\hat{M}_s$ with Lebesgue measure on $\hat{A}$.

Temporarily again, let $H=H_i$ and $P=MAN$. For each $\lambda=(\sigma, \nu) \in \Lambda$, $\sigma \in \hat{M}_s$, $\nu \in \hat{A}$, we form the unitary representation π^λ , obtained by inducing up to G the representation $m \mapsto e^{i\nu(\log a)} \sigma(m)$ of P . Actually these representations are "almost always" irreducible, but we shall not use that explicitly in what follows. The spaces $\mathcal{H}^\lambda$ on which the representations π^λ act are all infinite-dimensional separable Hilbert spaces; and so they are unitarily equivalent. Henceforth, we consider them to be so identified and we denote the common space (independent of λ or H_i) by $\mathcal{H}$.

Let μ be any Borel measure on the locally compact space $\Lambda = \bigcup_{i=1}^r \Lambda_i$. Then we may consider the Banach spaces $L_p(\Lambda, \mathcal{H}, \mu)$, $1 \leq p \leq \infty$, consisting of measurable operator-valued functions $F: \Lambda \rightarrow$ bounded linear operators on $\mathcal{H}$ with finite norm

$$\|F\|_p^p = \int_{\Lambda} \|F(\lambda)\|_p^p d\mu(\lambda), \quad 1 \leq p < \infty,$$

$$\|F\|_{\infty} = \operatorname{ess\,sup}_{\lambda \in \Lambda} \|F(\lambda)\|_{\infty}.$$

(We follow the terminology of [3a, §2] here for the most part.) In particular, we consider the Plancherel measure $\hat{\mu}$ computed by Harish-Chandra in [2, §12]. On Λ_i , $d\hat{\mu}(\lambda) = C_i(\sigma, \nu) d\lambda$ where the $C_i(\sigma, \nu)$ are certain continuous functions on Λ_i . (See [2] for more extensive information on the smoothness and growth properties of these functions.) Henceforth, we denote $L_p(\Lambda, \mathcal{H}, \hat{\mu})$ by $L_p(\Lambda)$.

Now define the Fourier transform $\mathcal{F}: L_1(G) \rightarrow L_\infty(\Lambda)$ as follows:

$$\mathcal{F}(f) = \hat{f}, \quad \hat{f}(\lambda) = \int_G f(g) \pi^\lambda(g) dg, \quad f \in L_1(G).$$

$\mathcal{F}$ is clearly a norm-decreasing linear mapping

$$(2.1) \quad \|\hat{f}\|_\infty \leq \|f\|_1.$$

It follows from Harish-Chandra's work that $\mathcal{F}$ extends from $L_1(G) \cap L_2(G)$ to an isometry of $L_2(G)$ into the Hilbert space $L_2(\Lambda)$:

$$(2.2) \quad \|f\|_2^2 = \|\hat{f}\|_2^2 = \int_\Lambda \|\hat{f}(\lambda)\|_2^2 d\hat{\mu}(\lambda).$$

(The mapping is essentially onto. We could render it thus if, in the definition of $L_2(\Lambda)$, we took into account the equivalences that occur amongst the π^λ .) A trivial application of [3a, Theorem 3] to (2.1) and (2.2) yields

$$(2.3) \quad \|\hat{f}\|_q \leq \|f\|_p, \quad f \in L_1(G) \cap L_p(G),$$

$1 < p < 2$, $1/p + 1/q = 1$. Hence $\mathcal{F}$ extends to a norm-decreasing linear mapping of $L_p(G)$ into $L_q(\Lambda)$. Specifically, if $f \in L_p(G)$, one chooses $f_n \in L_1(G) \cap L_p(G)$, $f_n \rightarrow f$ in L_p . It follows from (2.3) that $\hat{f}_n$ is Cauchy in $L_q(\Lambda)$. Then $\hat{f} = \mathcal{F}(f)$ is defined to be the $L_q(\Lambda)$ -limit of the sequence $\{\hat{f}_n\}$. It is easy to check that $\hat{f}$ is independent of the choice of f_n ; and that if f is also in $L_{p_1}(G)$, $1 \leq p_1 \leq 2$, then the resulting $\hat{f}$ in $L_{q_1}(\Lambda)$, $1/p_1 + 1/q_1 = 1$, differs from the previous $\hat{f} \in L_q(\Lambda)$ on at most a set of measure zero. Thus, given $f \in L_p(G)$, $1 \leq p \leq 2$, $\mathcal{F}(f) = \hat{f}$ is uniquely defined up to a set of measure zero.

Consider the following subsets of $L_\infty(\Lambda)$:

$$C(\Lambda) = \{F \in L_\infty(\Lambda): \lambda \rightarrow \|F(\lambda)\|_\infty \text{ is a (bounded) continuous function on } \Lambda\},$$

$$C_\infty(\Lambda) = \{F \in C(\Lambda): \text{the function } \lambda \rightarrow \|F(\lambda)\|_\infty \text{ vanishes at infinity}\}.$$

Recall the latter means that for each $\varepsilon > 0$, the set

$$\{\lambda \in \Lambda: \|F(\lambda)\|_\infty \geq \varepsilon\}$$

is compact.

LEMMA. For any $f \in L_1(G)$, we have $\hat{f} \in C_\infty(\Lambda)$.

PROOF. Let $f \in L_1(G)$. Then $\hat{f} \in L_\infty(\Lambda)$. We have to show that $\lambda \rightarrow \|\hat{f}(\lambda)\|_\infty$ is continuous and vanishes at infinity. Suppose $f \in C_0^\infty(G)$; then by [2, p. 544] the function $\lambda \rightarrow \text{Trace } \hat{f}(\lambda)$ has both these properties. It follows from [1, Theorem 2.3] that for $f \in L_1(G)$, $\lambda \rightarrow \|\hat{f}(\lambda)\|_\infty$ is a continuous function. Moreover, by [1, Corollary 1, p. 390], we get $\lim_{\lambda \rightarrow \infty} \|\hat{f}(\lambda)\|_\infty = 0$, $f \in L_1(G)$. This completes the proof.

Note. In order to apply Fell's results, we need to know that the group C^* -algebra of G has a dense subalgebra in which every element is boundedly represented [1, p. 389]. The existence of a "large compact subgroup" in G insures that this is true (see e.g. [4e, Lemma 2.2 and §6]).

3. The conditions. Let G be as in §2 and $1 \leq p \leq 2$.

(i) We say that G possesses the *Riemann-Lebesgue property* (with index p) if for every $f \in L_p(G)$ we have $\hat{f} \in C_\infty(\Lambda)$, i.e. there is $F \in C_\infty(\Lambda)$ such that $F = \hat{f}$ a.e.

(ii) We say that G possesses the *essential boundedness property* (with index p) if for every $f \in L_p(G)$ we have $\hat{f} \in L_\infty(\Lambda)$, i.e. there is $F \in L_\infty(\Lambda)$ such that $F = \hat{f}$ a.e.

(iii) Let U be a unitary representation of G . We say that U is *extendible to L_p* if there is a constant A_p such that for every $f \in L_1(G) \cap L_p(G)$, we have $\|U_f\|_\infty \leq A_p \|f\|_p$, $U_f = \int_G f(g)U(g) dg$ (see [3a, p. 55]).

(iv) We say that a family $\{U^\alpha\}_{\alpha \in \Omega}$ of unitary representations of G is *uniformly extendible to L_p* if there is a single constant A_p such that

$$\|U_f^\alpha\|_\infty \leq A_p \|f\|_p,$$

for every $f \in L_1(G) \cap L_p(G)$ and every $\alpha \in \Omega$.

(v) We say that the L_p *convolution theorem* is valid for G if there is a constant A_p such that $\|f * h\|_2 \leq A_p \|f\|_2 \|h\|_p$ for every $f \in L_2(G)$, $h \in L_p(G)$ (see [4c, p. 765]).

4. The main result.

THEOREM. Let G be a connected semisimple Lie group with finite center. Let $1 \leq p \leq 2$. Then the following are equivalent:

- (i) G possesses the *Riemann-Lebesgue property* (index p);
- (ii) G possesses the *essential boundedness property* (index p);
- (iii) The family $\{\pi^\lambda: \lambda \in \Lambda\}$ is *uniformly extendible to L_p* ;
- (iv) The *regular representation of G is extendible to L_p* ;
- (v) The L_p *convolution theorem is valid for G* .

PROOF. (i) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). By the hypothesis, we may consider the Fourier transform to be a linear map $\mathcal{F}: L_p(G) \rightarrow L_\infty(\Lambda)$. We show that the map has a closed graph. Suppose that the sequence $(f_n, \hat{f}_n)$ converges on the graph to $(f, F) \in L_p(G) \times L_\infty(\Lambda)$. Then

$$(4.1) \quad \|f_n - f\|_p \rightarrow 0$$

and

$$(4.2) \quad \|\hat{f}_n - F\|_\infty \rightarrow 0.$$

Applying (2.3) to (4.1) we obtain $\|f_n - f\|_q \rightarrow 0$. That is,

$$\int_{\Lambda} \|f_n(\lambda) - f(\lambda)\|_q^q d\hat{\mu}(\lambda) \rightarrow 0.$$

Therefore, we may choose a subsequence f_{n_j} such that $\|f_{n_j}(\lambda) - f(\lambda)\|_q \rightarrow 0$ a.e. But then

$$(4.3) \quad \|f_{n_j}(\lambda) - f(\lambda)\|_{\infty} \leq \|f_{n_j}(\lambda) - f(\lambda)\|_q \rightarrow 0 \quad \text{a.e.}$$

Finally, (4.2) implies that

$$(4.4) \quad \operatorname{ess\,sup}_{\lambda} \|f_n(\lambda) - F(\lambda)\|_{\infty} \rightarrow 0.$$

It follows from (4.3) and (4.4) that $F=f$ a.e. This is precisely what is needed in order to demonstrate that $\mathcal{F}$ has a closed graph.

By the closed graph theorem, $\mathcal{F}: L_p(G) \rightarrow L_{\infty}(\Lambda)$ is a *bounded* linear map, i.e. there is A_p such that

$$\operatorname{ess\,sup}_{\lambda} \|f(\lambda)\|_{\infty} \leq A_p \|f\|_p, \quad f \in L_p(G).$$

Hence, by the Lemma,

$$\|f(\lambda)\|_{\infty} \leq A_p \|f\|_p, \quad f \in L_1(G) \cap L_p(G),$$

for all $\lambda \in \Lambda$.

(iii) $\Rightarrow$ (v). This proof is fairly standard by now (see e.g. [3a, Theorem 9] or [4a, Theorem 4]). It is enough to consider $f \in L_1(G) \cap L_2(G)$, $h \in L_1(G) \cap L_p(G)$. Then

$$\begin{aligned} \|f * h\|_2^2 &= \int_{\Lambda} \|(f * h)^{\wedge}(\lambda)\|_2^2 d\hat{\mu}(\lambda) = \int_{\Lambda} \|f(\lambda)\hat{h}(\lambda)\|_2^2 d\hat{\mu}(\lambda) \\ &\leq \int_{\Lambda} \|f(\lambda)\|_2^2 \|\hat{h}(\lambda)\|_{\infty}^2 d\hat{\mu}(\lambda) \leq A_p^2 \|h\|_p^2 \int_{\Lambda} \|f(\lambda)\|_2^2 d\hat{\mu}(\lambda) \\ &= A_p^2 \|h\|_p^2 \|f\|_2^2. \end{aligned}$$

(v) $\Rightarrow$ (iv). Kunze and Stein have already observed [3a, p. 55] that these two statements are in fact different ways of saying the same thing.

(iv) $\Rightarrow$ (i). It is clear from the Plancherel formula (2.2) that the Hilbert space $L_2(G)$ decomposes into the direct integral $\int_{\Lambda}^{\oplus} \mathcal{H}^{\lambda} d\hat{\mu}(\lambda)$ via the Fourier transform. Moreover, the components of the regular representation under this decomposition are precisely π^{λ} , $\lambda \in \Lambda$. Therefore the second part of the proof of [3a, Theorem 11] shows that there is A_p such that

$$\operatorname{ess\,sup}_{\lambda} \|f(\lambda)\|_{\infty} \leq A_p \|f\|_p, \quad f \in L_1(G) \cap L_p(G).$$

But by continuity (i.e. by the Lemma), we obtain

$$(4.5) \quad \|\hat{f}(\lambda)\|_{\infty} \leq A_p \|f\|_p, \quad f \in L_1(G) \cap L_p(G), \lambda \in \Lambda.$$

Let $\lambda \in \Lambda$ be arbitrary but fixed and suppose $f \in L_p(G)$. Choose $f_n \in L_1(G) \cap L_p(G)$, $f_n \rightarrow f$ in L_p . Then $\{\hat{f}_n(\lambda)\}$ is a Cauchy sequence in the space $\mathcal{B}$ of all bounded linear operators on $\mathcal{H}$ (because of (4.5)). Let $\hat{f}(\lambda)$ be its limit. It is routine to see (by a proof analogous to the argument used in (ii) $\Rightarrow$ (iii)) that for any $f \in L_p(G)$, $\hat{f} = \hat{f}$ a.e. Thus $\hat{f} \in L_{\infty}(\Lambda)$ and we have a stronger version of (4.5), namely

$$(4.6) \quad \|\hat{f}(\lambda)\|_{\infty} \leq A_p \|f\|_p, \quad f \in L_p(G), \lambda \in \Lambda.$$

It only remains to show that $\hat{f} \in C_{\infty}(\Lambda)$.

Let $f \in L_p(G)$, $\varepsilon > 0$. Choose $h \in L_1(G) \cap L_p(G)$ such that $\|f - h\|_p < \varepsilon/2A_p$. Fix an arbitrary $\lambda_0 \in \Lambda$. We show continuity at λ_0 . If λ is sufficiently close to λ_0 , then

$$(4.7) \quad |\|\hat{h}(\lambda)\|_{\infty} - \|\hat{h}(\lambda_0)\|_{\infty}| < \varepsilon.$$

Moreover, using (4.6) and (4.7), we have

$$\begin{aligned} |\|\hat{f}(\lambda)\|_{\infty} - \|\hat{f}(\lambda_0)\|_{\infty}| &\leq |\|\hat{f}(\lambda)\|_{\infty} - \|\hat{h}(\lambda)\|_{\infty}| \\ &\quad + |\|\hat{h}(\lambda)\|_{\infty} - \|\hat{h}(\lambda_0)\|_{\infty}| \\ &\quad + |\|\hat{h}(\lambda_0)\|_{\infty} - \|\hat{f}(\lambda_0)\|_{\infty}| \\ &< 2 \sup_{\lambda} \|\hat{f}(\lambda) - \hat{h}(\lambda)\|_{\infty} + \varepsilon \leq 2\varepsilon. \end{aligned}$$

That is, $\|\hat{f}(\lambda)\|_{\infty}$ is close to $\|\hat{f}(\lambda_0)\|_{\infty}$ and we have proven continuity.

Finally, since $\|\hat{f}(\lambda) - \hat{h}(\lambda)\|_{\infty} < \varepsilon/2$ for all $\lambda \in \Lambda$, we have

$$(4.8) \quad \{\lambda \in \Lambda: \|\hat{f}(\lambda)\|_{\infty} \geq \varepsilon\} \subseteq \{\lambda \in \Lambda: \|\hat{h}(\lambda)\|_{\infty} \geq \varepsilon/2\}.$$

But $h \in L_1(G)$ implies (by the Lemma) that the right side of (4.8) is compact. Since $\lambda \rightarrow \|\hat{f}(\lambda)\|_{\infty}$ is continuous, the left side is closed, and therefore also compact. Because ε is arbitrary, we conclude that $\hat{f} \in C_{\infty}(\Lambda)$.

5. Remarks. (i) If G is compact, it is easy to see that the conditions of the Theorem are fulfilled for any $p \geq 1$ (since $L_p(G) \subseteq L_1(G)$). However, if G is noncompact, the L_2 convolution theorem is false (see [5]). For certain groups, these five conditions are true when $1 \leq p < 2$, and it has been conjectured that they hold for any G as in §2 (see [4d, §1]). This question is still open.

(ii) The specific groups for which the L_p convolution theorem has been substantiated have all been handled in essentially the same way; namely by an analytic continuation of the representations π^{λ} , $\lambda = (\sigma, \nu)$, to uniformly bounded representations depending on $\nu \in \mathfrak{a}_C^*$. This method has

only borne fruit when applied to a minimal parabolic subgroup—the case of nonminimal parabolics appears to be significantly more difficult. One is naturally led to the question: Can we obtain the L_p convolution theorem by some means other than analytic continuation? This was one of our main motivations in looking for equivalent conditions.

(iii) The set $C_\infty(\Lambda)$ is not a linear space (because the continuity of $\lambda \rightarrow \|F_1(\lambda)\|_\infty$ and $\lambda \rightarrow \|F_2(\lambda)\|_\infty$ does not imply the continuity of $\lambda \rightarrow \|F_1(\lambda) + F_2(\lambda)\|_\infty$). However, in those groups for which analytic continuation has been carried out, it is possible to show that for $f \in L_1(G)$, $\hat{f}(\lambda)$ is continuous in the uniform topology; i.e.

$$\lim_{\lambda \rightarrow \lambda_0} \|\hat{f}(\lambda) - \hat{f}(\lambda_0)\|_\infty = 0.$$

It seems likely then that one can replace $C_\infty(\Lambda)$ in the Theorem by the linear space $\mathcal{C}_\infty(\Lambda) = \{F \in L_\infty(\Lambda) : \lambda \rightarrow F(\lambda) \text{ is continuous in the uniform topology and } \lim_{\lambda \rightarrow \infty} \|F(\lambda)\|_\infty = 0\}$.

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