A CHARACTERIZATION OF STRONGLY MEASURABLE PETTIS INTEGRABLE FUNCTIONS

J. J. UHL, JR.¹

ABSTRACT. **THEOREM.** *Let $\mathfrak{X}$ be a Banach space and (Ω, Σ, μ) be a finite measure space. A strongly measurable $f: \Omega \rightarrow \mathfrak{X}$ is Pettis integrable if and only if there exists a Young's function Φ with $\lim_{t \rightarrow \infty} \Phi(t)/t = \infty$ such that $x^*f \in L^1(\mu)$ for all $x^* \in \mathfrak{X}^*$.*

In 1969, Brooks [1] gave a characterization of strongly measurable Pettis integrable functions by use of the Orlicz-Pettis theorem. The purpose of this note is to give another characterization in terms of Orlicz spaces. Throughout this note $\mathfrak{X}$ is a Banach space with continuous dual $\mathfrak{X}^*$. (Ω, Σ, μ) is a fixed finite measure space. $f: \Omega \rightarrow \mathfrak{X}$ is called strongly measurable if it is the μ -almost everywhere limit of simple functions of the form $\sum_{i=1}^n x_i \chi_{E_i}$ where $x_i \in \mathfrak{X}$, $E_i \in \Sigma$ and χ_{E_i} is the characteristic or indicator function of E_i . A strongly measurable $f: \Omega \rightarrow \mathfrak{X}$ is called Pettis integrable if $x^*f \in L^1(\mu)$ for all $x^* \in \mathfrak{X}^*$ and for each $E \in \Sigma$ there is $x_E \in \mathfrak{X}$ satisfying $x^*(x_E) = \int_E x^*f d\mu$ for all $x^* \in \mathfrak{X}^*$. A Young's function is an increasing convex function Φ defined on $[0, \infty)$ such that $\Phi(0) = 0$. The Orlicz space $L^\Phi(\mu)$ is the space of all (equivalence classes of) functions f such that $\int_\Omega \Phi(|f|/k) d\mu < \infty$ for some $k > 0$.

The following theorem gives the promised characterization of Pettis integrable strongly measurable functions. In a sense this theorem is an analogue of LaVallée Poussin's theorem for vector valued functions.

THEOREM. *Let $f: \Omega \rightarrow \mathfrak{X}$ be strongly measurable. f is Pettis integrable with respect to μ if and only if there exists a Young's function Φ such that*

- (i) $\lim_{t \rightarrow \infty} \Phi(t)/t = +\infty$, and
- (ii) $x^*f \in L^\Phi(\mu)$ for each $x^* \in \mathfrak{X}^*$.

PROOF (Sufficiency). Suppose $x^*f \in L^\Phi(\mu)$ for all $x^* \in \mathfrak{X}^*$. Then if $T: \mathfrak{X}^* \rightarrow L^\Phi$ is defined by $T(x^*) = x^*f$, T is linear. Furthermore T is closed since if $x_n^* \rightarrow x^*$ in $\mathfrak{X}^*$ and $T(x_n^*) \rightarrow g$ in $L^\Phi(\mu)$ norm, then $x^*f = g$ a.e. $[\mu]$ because $x_n^*f \rightarrow x^*f = g$ pointwise almost everywhere. According to Banach's closed graph theorem, T is continuous. Hence if $g \in L^\Psi$ where Ψ is

Received by the editors July 6, 1971.

AMS 1969 subject classifications. Primary 2850, 3035.

¹ Supported in part by National Science Foundation grant GP-28577.

© American Mathematical Society 1972

complementary to Φ , and $S(g)$ is defined on $\mathfrak{X}^*$ by

$$S(g)(x^*) = \int_{\Omega} g x^* f d\mu, \quad x^* \in \mathfrak{X}^*,$$

then S is linear in g . Moreover for $x^* \in \mathfrak{X}^*$,

$$|S(g)(x^*)| = \left| \int_{\Omega} g x^* f d\mu \right| \leq \|g\|_{\Psi} \|x^* f\|_{\Phi} \leq \|g\|_{\Psi} \|x^*\| \|T\|.$$

Hence $S(g) \in \mathfrak{X}^{**}$ and $S(g)$ is the "Gelfand" integral of gf .

Next we shall show that S actually maps bounded functions in L^{Ψ} into the closed subspace $\mathfrak{X}$ of $\mathfrak{X}^{**}$. To this end, set $E_n = \{\omega : \|f(\omega)\| \leq n\}$. If g is supported in one of the E_n 's, say E_{n_0} , consider

$$S(g)(x^*) = \int_{\Omega} g x^* f d\mu, \quad x^* \in \mathfrak{X}^*.$$

Since f is bounded on E_{n_0} , gf is actually Bochner integrable. Hence

$$S(g)(x^*) = x^* \left(\text{Bochner} - \int_{\Omega} gf d\mu \right);$$

i.e. $S(g) = \text{Bochner} - \int_{\Omega} gf d\mu$ for g supported inside one of the E_n 's. Now condition (i) implies $\Psi(t) < \infty$ when $t < \infty$. Hence by [4, Theorem 10.3, p. 87], $\lim_{\mu(E) \rightarrow 0} \|g\chi_E\|_{\Psi} = 0$ for all g in the closed subspace of L^{Ψ} determined by the simple functions. Since $E_n \uparrow \Omega$, the continuity of S implies $S(g) = \lim_n S(g\chi_{E_n})$ strongly in $\mathfrak{X}^{**}$ for all g in the closed subspace of L^{Ψ} determined by the simple functions. In particular, $S(\chi_E) = \lim_n S(\chi_{E \cap E_n}) = \lim_n \text{Bochner} - \int_{\Omega} \chi_E \chi_{E_n} f d\mu \in \mathfrak{X}$. Hence $S(\chi_E)(x^*) = \int_{\Omega} \chi_E x^* f d\mu$ and f is Pettis integrable.

(Necessity). Suppose f is Pettis integrable. By a theorem of Pettis [6, Corollary 2.51, p. 284], $\lim_{\mu(E) \rightarrow 0} \int_E |x^* f| d\mu = 0$ uniformly for x^* in the unit ball of $\mathfrak{X}^*$. Also $\sup_{\|x^*\| \leq 1} \|x^* f\|_{L^1} < \infty$. An appeal to LaVallée Poussin's theorem [5, Theorem 22, p. 19] establishes the existence of a convex function Φ defined on $[0, \infty)$ which is nonnegative, increasing, convex and satisfying $\lim_{t \rightarrow \infty} \Phi(t)/t = +\infty$ such that

$$\sup_{\|x^*\| \leq 1} \int_{\Omega} \Phi(|x^* f|) d\mu < \infty.$$

By setting $\Phi_1(t) = \Phi(t) - \Phi(0)$, one finds that Φ_1 is a Young's function satisfying (i) and $x^* f \in L^{\Phi_1}(\mu)$ for all $x^* \in \mathfrak{X}^*$. This completes the proof of the theorem.

REFERENCES

1. James K. Brooks, *Representations of weak and strong integrals in Banach spaces*, Proc. Nat. Acad. Sci. U.S.A. **63** (1969), 266–270. MR **43** #266.
2. N. Dunford and J. T. Schwartz, *Linear operators. I: General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.
3. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R.I., 1957. MR **19**, 664.
4. M. A. Krasnosel'skiĭ and Ja. B. Rutickiĭ, *Convex functions and Orlicz spaces*, GITTL, Moscow, 1958; English transl., Noordhoff, Groningen, 1961. MR **21** #5144; MR **23** #A4016.
5. Paul A. Meyer, *Probability and potentials*, Blaisdell, Waltham, Mass., 1966. MR **34** #5119.
6. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 227–304.
7. A. C. Zaanen, *Linear analysis, measure and integral, Banach and Hilbert space, linear integral equations*, Interscience, New York; North-Holland, Amsterdam; Noordhoff, Groningen, 1953. MR **15**, 878.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801