ON RECOGNISING CERTAIN ONE RELATION PRESENTATIONS

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ABSTRACT. An algorithm is described to determine, given a finite presentation Π of a one relation group G whose commutator subgroup is finitely generated, whether or not Π is a one relation presentation.

1. Introduction. Let k, l be integers different from 0, ± 1 . It was shown in [2] that if r, s are integers satisfying $(r, s) = (r, k) = (s, l) = 1$, then the group $\langle x_1, x_2; x_1^k = x_2^l \rangle$ has presentation

$$\Pi = \langle x_1, x_2; x_1 x_2 x_1^{-l} x_2^{-k} = 1, x_1^k x_2 x_1^{-k} x_2^{-l} = 1, x_1^{sk} x_2^{-rl} = 1 \rangle.$$

In the study of these groups, the problem arises of determining for which values of r and s the presentation Π is (in the notation of [2]) a one relation presentation. The object of this note is to prove a result which has as a consequence the fact that, given k, l, r and s , it can be determined if the corresponding presentation Π is a one relation presentation.

A group G is said to be a one relation group if G has a presentation with a single defining relation. Our result is the following:

THEOREM. *Let G be a one relation group with finitely generated commutator subgroup G' , and let G be given by a presentation*

$$\Pi = \langle a_1, \dots, a_m; R_1 = 1, \dots, R_n = 1 \rangle.$$

Then it can be determined if Π is a one relation presentation.

2. Proof of the Theorem.

We shall need the following lemma.

LEMMA. *Let*

$$G = \langle b_1, b_2; V_1 = 1, \dots, V_n = 1 \rangle,$$

where the exponent sum of b_1 in each V_i is zero, and G is nonabelian with G' finitely generated. Then $\{b_1, b_2\}$ is a one relation generating set of G if and only if there exists a positive integer t such that the normal subgroup N

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of G generated by b_2 is freely generated by the set

$$\{b_2, b_1 b_2 b_1^{-1}, \dots, b_1^{t-1} b_2 b_1^{1-t}\}.$$

PROOF. Suppose that such an integer t exists. We put $x_i = b_1^i b_2 b_1^{-i}$ ($1 \leq i \leq t-1$). Since conjugation by b_1 induces an automorphism of N , and the infinite cyclic group G/N is generated by the image of b_1 under the natural homomorphism from G to G/N , it follows that G has a presentation

$$\langle b_1, b_2, x_1, \dots, x_{t-1}; b_1 b_2 b_1^{-1} = x_1, \dots, b_1 x_{t-2} b_1^{-1} = x_{t-1}, \\ b_1 x_{t-1} b_1^{-1} = w(b_2, x_1, \dots, x_{t-1}) \rangle$$

on its generating set $\{b_1, b_2, x_1, \dots, x_{t-1}\}$. Using the obvious Tietze transformations to remove the generators $x_1, \dots, x_{t-1}$, we see that G can be presented on $\{b_1, b_2\}$ by

$$\langle b_1, b_2; b_1^t b_2 b_1^{-t} = w(b_2, b_1 b_2 b_1^{-1}, \dots, b_1^{t-1} b_2 b_1^{1-t}) \rangle.$$

Thus $\{b_1, b_2\}$ is a one relation generating set of G . The converse follows easily from the proof of Theorem 1 of [3].

To prove the theorem, we first observe that if $m=1$ then Π is a one relation presentation, while if $m>2$ then, by Theorem 1 of [3], Π is not such a presentation. Thus we can assume that $m=2$.

From Π we can obtain, by Theorem 3.5 of [1], a presentation

$$\Pi_1 = \langle b_1, b_2; V_1 = 1, \dots, V_n = 1 \rangle$$

of G which is N -equivalent (in the notation of [2]) to Π , and such that

$$(1) \quad V_i = b_1^{d_i} Q_i \quad (i = 1, 2)$$

and

$$(2) \quad V_j = Q_j \quad (2 < j \leq n),$$

where $d_i \geq 0$ ($i=1, 2$), d_2 divides d_1 , and Q_i, Q_j belong to the commutator subgroup of the free group $F(b_1, b_2)$ on $\{b_1, b_2\}$. Since Π and Π_1 are N -equivalent, it follows that Π is a one relation presentation if and only if the same is true of Π_1 .

We now enumerate recursively the presentations of G , beginning with Π_1 , until we obtain a one relation presentation $\Pi_2 = \langle x_1, \dots, x_s; R=1 \rangle$, where $s \leq 2$, and the exponent sum of x_1 in R is zero if $s=2$. Let G_1, G_2 be the isomorphic copies of G determined by Π_1, Π_2 respectively. In the process of obtaining Π_2 we can also find words $u_1(x_1, \dots, x_s), u_2(x_1, \dots, x_s)$ such that the homomorphism θ from $F(b_1, b_2)$ to $F(x_1, \dots, x_s)$ given by $\theta b_i = u_i$ ($i=1, 2$) induces an isomorphism $\theta: G_1 \rightarrow G_2$.

Now if $s=1$ and R is nontrivial, then Π_1 is not a one relation presentation, since no single relation on b_1, b_2 can imply that G_1 is a finite cyclic group, while if $s=1$ and R is trivial then Π_1 is a one relation presentation, since then G_1 is infinite cyclic and so, from the conditions on the V_i , we see that $b_2=1$. Thus we can assume that $s=2$.

If G is abelian, then, by Problem 21, p. 283 of [1], either R is a primitive element of $F(x_1, x_2)$, or $R^{\pm 1}$ is conjugate in $F(x_1, x_2)$ to $x_1x_2x_1^{-1}x_2^{-1}$, and in either case Π_1 is a one relation presentation. Now using the algorithm of [4], it can be decided if R is primitive, and it can also be easily decided if $R^{\pm 1}$ is conjugate to $x_1x_2x_1^{-1}x_2^{-1}$. Thus we can assume that G is nonabelian.

Now G/G' is infinite, and so the exponent sum of b_1 in each relator V_i of Π_1 must be zero. Let N_1, N_2 be the normal subgroups generated respectively by b_2, x_2 in G_1 and G_2 . By Corollary 2 of Theorem 1 of [3] the exponent sum of x_2 in R is nonzero, since the exponent sum of x_1 in R is zero. Thus if N is a normal subgroup of G_2 such that G_2/N is infinite cyclic, then $x_2 \in N$ (since x_2 has finite order in G_2/N) and so $N_2 \subseteq N$. Now G_2/N_2 is infinite cyclic, and so $N=N_2$. Hence $\theta N_1=N_2$ since G_1/N_1 is infinite cyclic. From the results of [3] we have that N_2 is a finitely generated free group, and the rank t of N_2 can be found. Thus N_1 is free of rank t . It follows from the Lemma that Π_1 is a one relation presentation if and only if $\{b_2, b_1b_2b_1^{-1}, \dots, b_1^{t-1}b_2b_1^{1-t}\}$ is a free generating set of N_1 . This will be the case if and only if $\{\theta b_2, \theta(b_1b_2b_1^{-1}), \dots, \theta(b_1^{t-1}b_2b_1^{1-t})\}$ is a free generating set of N_2 . This can be decided by expressing the elements of this set in terms of the free generating set $\{x_2, x_1x_2x_1^{-1}, \dots, x_1^{t-1}x_2x_1^{1-t}\}$ of N_2 , and then applying Nielsen transformations to the resulting set to obtain a Nielsen reduced set. This proves the theorem.

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