CONTRACTIVE PROJECTIONS IN CONTINUOUS FUNCTION SPACES

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ABSTRACT. Let $C(K)$ be the Banach space of real-valued continuous functions on a compact Hausdorff space with the supremum norm and let X be a closed subspace of $C(K)$ which separates points of K . Necessary and sufficient conditions are given for X to be the range of a projection of norm one in $C(K)$. It is shown that the form of a projection of norm one is determined by a real-valued continuous function which is defined on a subset of K and which satisfies conditions imposed by X . When there is a projection of norm one onto X , it is shown that there is a one-to-one correspondence between the continuous functions which satisfy the conditions imposed by X and the projections of norm one onto X .

Using well-known results of Nachbin, Goodner and Kelley (see [2, p. 2]), it is easily demonstrated that if S is an extremely disconnected compact Hausdorff space, then X , a closed subspace of $C(S)$, is the range of a norm one projection in $C(S)$ iff X itself is isometric to the continuous functions on an extremely disconnected compact Hausdorff space. The main purpose of this research is to give necessary and sufficient conditions for a closed separating subspace X of $C(K)$, K a compact Hausdorff space, to be the range of a norm one projection in $C(K)$. Theorem 3.1 supplies these conditions. Proposition 1.2 shows that the form of each contractive projection onto X can be given in terms of a real-valued continuous function, determined by the projection, which is defined on a subset of K . This continuous function satisfies certain conditions which depend only on the subspace X . Theorem 3.1 also shows that when there is a contractive projection onto X there is a one-to-one correspondence between contractive projections onto X and the continuous functions which satisfy these conditions imposed by X . We give another proof to the fact, proved in [3], that the Banach space Y is isometric to the range of a contractive

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projection in some space $C(Q)$, Q compact Hausdorff, iff Y is isometric to $C_\sigma(L)$ for some compact Hausdorff space L and involutive homeomorphism σ of L . We also extend some results of [4].

Throughout this paper K is to be a compact Hausdorff space and $C(K)$ is to be the Banach space of real-valued continuous functions on K endowed with the supremum norm. A *projection* in a Banach space Y is a bounded linear operator P on Y such that $P^2 = P$; a *contractive projection* is a projection of norm one. Subspace is to mean closed subspace.

A subspace Y of $C(K)$ is said to be *separating* if, for each $x \neq y$ in K , there is an f in Y such that $f(x) \neq f(y)$. An *involutive homeomorphism* T of a topological space is a homeomorphism such that T^2 is the identity map. If δ is an involutive homeomorphism of K , then $C_\delta(K)$ is the space $\{f \text{ in } C(K) : f(x) = -f(\delta(x)) \text{ for all } x \text{ in } K\}$ endowed with the supremum norm. By an *isometry* T from the Banach space Y onto the Banach space Z we mean a bounded linear map with bounded inverse such that $\|T(f)\|_Z = \|f\|_Y$ for all f in Y ; in this case Y and Z are said to be *isometric*. If Y is a Banach space, B_Y , the unit ball of Y , is $\{y \text{ in } Y : \|y\| \leq 1\}$. If x is in K , $\pm x'$ is the linear functional in $C(K)^*$ such that $\pm x'(f) = \pm f(x)$. It is shown in [1, p. 441] that for any closed subspace Y of $C(K)$, the extreme points of B_Y^* are of the form $\pm x'|_Y$. A point x in K is a *double point* of X , a separating subspace, if there is a y in K such that for all f in X , $f(x) = -f(y)$. A point of K is a *single point* if it is not a double point. By *single* (respectively *double*) *extreme point* we mean a single (respectively double) point x such that $x'|_X$ is an extreme point of B_X^* . The double points are closed and the map $\bar{\sigma}$, defined on the double points by $\bar{\sigma}(x) = y$ where $f(y) = -f(x)$ for all f in X , is an involutive homeomorphism of the double points. Now let S be the closure in K of the single extreme points; let D be the closure of the double extreme points and let σ be $\bar{\sigma}|_D$. The map σ is an involutive homeomorphism of D since $\bar{\sigma}$ maps double extreme points into double extreme points.

Throughout this paper X is to denote a separating subspace of $C(K)$ and S , D and σ are to be as stated above.

1. Necessary conditions for the existence of a contractive projection.

PROPOSITION 1.1. *Let P be a contractive projection in $C(K)$ onto X . For all x in S , $P^*(x'|_X) = x'$; for all x in D , there is a t , $0 \leq t \leq 1$, such that*

$$P^*(x'|_X) = tx' - (1 - t)(\sigma(x))'.$$

PROOF. Let x be in K such that $x'|_X$ is an extreme point of B_X^* . Let $E_x = \{x^* \text{ in } B_{C(K)^*} : x^*|_X = x'|_X\}$. E_x is convex and w^* -closed since $E_x = \bigcap_{f \in X} \{x^* \text{ in } B_{C(K)^*} : x^*(f) = f(x)\}$. Hence E_x is the closed convex hull of its extreme points. If x^* is an extreme point of E_x , it is an extreme point of

$B_{C(K)^*}$. This is true since if $x^* = \frac{1}{2}y^* + \frac{1}{2}z^*$ where y^* and z^* are in $B_{C(K)^*}$, then $x'|_X = x^*|_X = \frac{1}{2}y^*|_X + \frac{1}{2}z^*|_X$. Since $x'|_X$ is an extreme point of B_{X^*} , $y^*|_X = z^*|_X = x^*|_X = x'|_X$ and hence y^* and z^* are in E_x . But since x^* is an extreme point of E_x , we have $x^* = y^* = z^*$ and x^* is an extreme point of $B_{C(K)^*}$. It follows that $x^* = \pm y'$ for some y in K . Now using the fact that X separates points and $x^*|_X = x'$, we have that if $x^* = y'$, then $x = y$ and if $x^* = -y'$, $y = \sigma(x)$. Hence if x is a single extreme point, $E_x = \{x'\}$; and if x is a double extreme point, E_x is the closed convex hull of x' and $-y' = -(\sigma(x))'$, in which case E_x consists of exactly those y^* such that $y^* = tx' + (1-t)(-(\sigma(x))') = tx' - (1-t)(\sigma(x))'$ for some t , $0 \leq t \leq 1$. Now, $P^*(x'|_X)$ is in E_x . Therefore, if x is a single extreme point, $P^*(x'|_X) = x'$; and if x is a double extreme point, there is a t , $0 \leq t \leq 1$, such that $P^*(x'|_X) = tx' - (1-t)(\sigma(x))'$. The conclusion of the proposition now follows by noticing that if $\{x_\alpha\}$ is a net converging to x in K , then $P^*(x'_\alpha|_X)$ converges to $P^*(x'|_X)$.

PROPOSITION 1.2. *If there is a contractive projection P in $C(K)$ onto X , then $S \cap \sigma(S \cap D) = \emptyset$ and there is a continuous function c from $S \cup D$ into $[0, 1]$ (the unit interval), uniquely determined by P , such that for all f in $C(K)$:*

$$P(f)(x) = c(x)f(x) = f(x) \quad \text{for } x \text{ in } S;$$

and

$$P(f)(x) = c(x)f(x) - (1 - c(x))f(\sigma(x)) \quad \text{for } x \text{ in } D.$$

Furthermore, $c(x) = 1$ for x in S and $c(x) + c(\sigma(x)) = 1$ for x in D .

PROOF. For all x in $S \cup D$, define $c(x) = \frac{1}{2}(P(1_K)(x) + 1)$, where 1_K is the function identically one on K . Since P is a contractive projection, $-1 \leq P(1_K)(x) \leq 1$; therefore c has the desired range. By Proposition 1.1, for x in S , $P(f)(x) = f(x)$ for all f in $C(K)$ and hence $P(1_K)(x) = 1$ and $c(x) = 1$. Also by Proposition 1.1, for x in D , there is a t such that $P^*(x'|_X) = tx' - (1-t)(\sigma(x))'$. Hence, $P(1_K)(x) = 2t - 1$ and $c(x) = t$. Therefore for all f in $C(K)$ and x in D , $P(f)(x) = c(x)f(x) - (1 - c(x))f(\sigma(x))$. Since for all f in X , $f(x) = -f(\sigma(x))$, we have that for x in D , $c(x) + c(\sigma(x)) = \frac{1}{2}[P(1_K)(x) + 1 + P(1_K)(\sigma(x)) + 1] = 1$. This property of c along with the property that for x in S , $c(x) = 1$, yields that $S \cap \sigma(S \cap D) = \emptyset$.

PROPOSITION 1.3. *If there is a contractive projection P in $C(K)$ onto X , then X is isometric by restriction to $S \cup D$ to a subspace Y of $C(S \cup D)$ where $Y = \{f \text{ in } C(S \cup D) : f(x) = -f(\sigma(x)) \text{ for } x \text{ in } D\}$.*

PROOF. The restriction map is linear. It preserves norm since for f in X , $\|f\|_X = \sup_{x \in \text{Ext } B_{X^*}} |x^*(f)|$; but since every extreme point of B_{X^*} is of the form $\pm x'|_X$ for some x in $S \cup D$, we have $\|f\|_X = \sup_{x \in S \cup D} |f(x)| = \|f|_{S \cup D}\|_{C(S \cup D)}$. We need only show this restriction map is onto Y . Given

f in Y choose $\bar{f}$ in $C(K)$ such that $\bar{f}|_{S \cup D} = f$. Now $P(\bar{f})$ is in X and $P(\bar{f})|_{S \cup D} = f$ since, by Proposition 1.2, if x is in S , $P(\bar{f})(x) = \bar{f}(x) = f(x)$ and if x is in D , $P(\bar{f})(x) = c(x)\bar{f}(x) - (1 - c(x))\bar{f}(\sigma(x)) = f(x)$.

COROLLARY 1.4. *Assume the hypotheses of Proposition 1.3.*

(a) *If for all x in D , $P(1_K)(x) \neq 1$, then X is isometric by restriction to $S \cup D$ to $C(S) \oplus_{C_\sigma}(D)$ with the supremum norm sum. If $\|I - P\| < 2$ where I is the identity operator on $C(K)$, then $P(1_K) \neq 1$ for x in D .*

(b) *If $P(1_K) \neq 0$ for all x in D , then there is a closed subset Q of K such that X is isometric by restriction to Q to $C(Q)$.*

PROOF. (a) If $P(1_K) \neq 1$ for all x in D , by Proposition 1.2, $c(x) \neq 1$ for all x in D and $S \cap D = \emptyset$. Now we may use Proposition 1.3.

(b) If $P(1_K)(x) \neq 0$, then $c(x) \neq \frac{1}{2}$ for x in D and, using Proposition 1.2, we have D is the disjoint union of two closed homeomorphic sets, A and $\sigma(A)$, where $A = \{x \text{ in } D : c(x) > \frac{1}{2}\}$. Let $Q = S \cup A$.

COROLLARY 1.5. *Let Y be any closed subspace of $C(K)$. If there is a contractive projection P in $C(K)$ onto Y , then Y is isometric to $C_\delta(L)$ where L is a compact Hausdorff space and δ an involutive homeomorphism of L .*

PROOF. We may assume Y is a separating subspace of $C(K)$ since Y is isometric to a separating subspace of $C(Q)$ where Q is the quotient space of K obtained by equating points in K on which all functions in Y agree. $C(Q)$ may be considered as a subspace of $C(K)$ and hence $P|_{C(Q)}$ is a contractive projection in $C(Q)$ onto Y . Therefore, by Proposition 1.3, there are sets S , D and an involutive homeomorphism σ of D such that $S \cap \sigma(S \cap D) = \emptyset$ and Y is isometric to $\{f \text{ in } C(S \cup D) : f(x) = -f(\sigma(x)) \text{ for all } x \text{ in } D\}$. We now show this space is isometric to $C_\delta(L)$. Let L be the set of points consisting of $S \cup D$ together with the set A which is a copy of $S - D$, where a point x in $S - D$ is identified with $\bar{x}$ in A . Now define δ as follows: $\delta(x) = \sigma(x)$ for x in D , $\delta(x) = \bar{x}$ for x in $S - D$, and $\delta(\bar{x}) = x$ for $\bar{x}$ in A . A basis for the topology on L is as follows: An open neighborhood of a point x in $S \cup D - \sigma(S \cap D)$ is of the form $N_x - \sigma(S \cap D)$ where N_x is an open neighborhood of x in $S \cup D$; an open neighborhood of a point y in $\sigma(S \cap D) \cup A$ is of the form $\delta(N_x)$ where $\delta(x) = y$ and N_x is an open neighborhood of x in $S \cup D$. With this topology L is a compact Hausdorff space and δ is an involutive homeomorphism of L such that $\delta|_D = \sigma$. Now the map extending each function f in $\{f \text{ in } C(S \cup D) : f(\sigma(x)) = -f(x)\}$ to $\bar{f}$ in $C_\delta(L)$ by $\bar{f}(\bar{x}) = -f(\delta(\bar{x}))$ is an isometry onto $C_\delta(L)$. Hence Y is isometric to $C_\delta(L)$.

EXAMPLE 1.6. Consider $C[0, 1]$, the continuous real-valued functions on the unit interval. Let $Y = \{f \text{ in } C[0, 1] : f(0) = -f(1)\}$. The single extreme points of Y are the points of the open interval $(0, 1)$; the double

extreme points are the points $\{0, 1\}$. Hence $D = \{0, 1\}$ and $S = [0, 1]$. The involutive homeomorphism of D is $\sigma(0) = 1$. Now, since $S \cap \sigma(S \cap D) = D \neq \emptyset$, there is no contractive projection in $C[0, 1]$ onto Y by Proposition 1.2.

REMARKS. If δ is any involutive homeomorphism of K we observe that $C_\delta(K)$ is the range of the contractive projection P in $C(K)$ where $P(f)(x) = \frac{1}{2}f(x) - \frac{1}{2}f(\delta(x))$ for all f in $C(K)$ and all x in K . This observation along with Corollary 1.5 gives another proof of the following result of [3]: A Banach space Y is isometric to the range of a contractive projection in some $C(Q)$ iff Y is isometric to some $C_\delta(L)$. This result does not help us determine those subspaces of $C(K)$ which are the range of a contractive projection in $C(K)$. Example 1.6 gives a subspace Y which, although not the range of a contractive projection in $C[0, 1]$, is isometric to $C_\sigma(S_1)$ where S_1 is the unit circle and σ is the antipodal map.

In [4] it is shown that a subspace Y of $C(K)$ is isometric to $C(L)$, L a compact Hausdorff space, if Y is the range of a contractive projection in $C(K)$ and Y has the following property: Y , considered as a separating subspace of $C(Q)$, where Q is the quotient space of K obtained by equating points of K on which all functions in Y agree, has no double extreme points. This result follows from Corollary 1.4 since Y can be considered as the range of a contractive projection in $C(Q)$ with $D = \emptyset$.

2. Sufficient conditions for the existence of a contractive projection.

PROPOSITION 2.1. *There is a continuous function from $S \cup D$ into $[0, 1]$ such that $c(x) + c(\sigma(x)) = 1$ for x in D and $c(x) = 1$ for x in S iff $S \cap \sigma(S \cap D) = \emptyset$.*

PROOF. If such a c exists, $c(\sigma(x)) = 0$ for all x in $S \cap D$; hence $S \cap \sigma(S \cap D) = \emptyset$. Suppose $S \cap \sigma(S \cap D) = \emptyset$. If $S \cap D = \emptyset$, $c(x) = \frac{1}{2}$ for x in D and $c(x) = 1$ for x in S has the desired properties; so assume $S \cap D \neq \emptyset$. There are open, disjoint sets u and v in D with the relative topology such that $S \cap D$ is contained in u and $\sigma(S \cap D)$ is contained in v . For each x in $S \cap D$, there is an open set u_x containing x and contained in u with $\sigma(u_x)$ contained in v . Let $u' = \bigcup_{x \in S \cap D} u_x$ and $v' = \sigma(u')$. Now u' and v' are disjoint open sets in D . Choose a continuous function d from D into $[0, 1]$ such that $d|_{\sigma(S \cap D)} = 0$ and $d(x) = 1$ for x not in v' . We have $d(x) = 1$ for x in u' and $d(x) + d(\sigma(x)) \neq 0$ for x in D . Define $c(x) = d(x)/(d(x) + d(\sigma(x)))$ for all x in D and $c(x) = 1$ for x in S . Now c has the desired properties.

PROPOSITION 2.2. *If $S \cap \sigma(S \cap D) = \emptyset$ and X is isometric by restriction to $S \cup D$ to $Y = \{f \text{ in } C(S \cup D) : f(x) = -f(\sigma(x)) \text{ for } x \text{ in } D\}$, then there is a contractive projection in $C(K)$ onto X .*

PROOF. Let R be the restriction isometry from X onto Y . By Proposition 2.1 there is a continuous function c from $S \cup D$ into $[0, 1]$ such that $c(x)=1$ for x in S and $c(x)+c(\sigma(x))=1$ for x in D . Define a map T_c from $C(K)$ onto Y such that for f in $C(K)$, $T_c f(x)=f(x)$ for x in S and $T_c(f)(x)=c(x)f(x)-(1-c(x))f(\sigma(x))$ for x in D . For each f in $C(K)$, $T_c(f)$ is continuous on $S \cup D$ since c is continuous; also for x in D ,

$$\begin{aligned} T_c(f)(\sigma(x)) &= c(\sigma(x))f(\sigma(x)) - (1-c(\sigma(x)))f(\sigma^2 x) \\ &= (1-c(x))f(\sigma(x)) - c(x)f(x) = -T_c(f)(x). \end{aligned}$$

Hence T_c is a map into Y ; T_c is linear. We have $\|T_c\|=1$ since for f in X , $T_c(f)=f|_{S \cup D}=R(f)$, and for all f in $C(K)$, $|T_c(f)(x)|=|f(x)|$ for x in S , and for x in D , $|T_c(f)(x)| \leq |c(x)+ (1-c(x))| \|f\| = \|f\|$. Also T_c is onto Y since for f in Y , $R^{-1}(f)$ is in X and $T_c(R^{-1}(f))=R(R^{-1}(f))=f$. Now define $P_c=R^{-1}T_c$. P_c is a linear operator in $C(K)$ with range X and $\|P_c\| \leq \|R^{-1}\| \cdot \|T_c\|=1$. For f in X , $P_c(f)=R^{-1}T_c(f)=R^{-1}(R(f))=f$. Hence P_c is a contractive projection $C(K)$ onto X .

3. The main theorem.

THEOREM 3.1. *Let X be a closed separating subspace of $C(K)$. Let S (respectively D) be the closure in K of the single (respectively double) extreme points of X . Let σ be the involutive homeomorphism of D such that for all f in X , $f(\sigma(x))=-f(x)$ for all x in D . There is a contractive projection in $C(K)$ onto X iff $S \cap \sigma(S \cap D) = \emptyset$ and X is isometric by restriction to $S \cup D$ to $Y = \{f \text{ in } C(S \cup D) : f(x) = -f(\sigma(x)) \text{ for all } x \text{ in } D\}$. Furthermore, if X satisfies the above conditions, there is a one-to-one correspondence between the contractive projections onto X and the continuous functions c from $S \cup D$ into $[0, 1]$ where $c(x)+c(\sigma(x))=1$ for x in D and $c(x)=1$ for x in S . This correspondence is given as follows: For each contractive projection P onto X , $\frac{1}{2}(P(1_K)(x)+1)$ is a continuous function which when restricted to $S \cup D$ has the desired properties; and for each function c with the desired properties, there is a contractive projection, P_c , onto X such that for x in $S \cup D$, $c(x) = \frac{1}{2}(P_c(1_K)(x)+1)$ with $P_c=R^{-1}T_c$ where R is the restriction isometry and T_c is such that for all f in $C(K)$, $T_c(f)(x)=f(x)$ for x in S and $T_c(f)(x)=c(x)f(x)-(1-c(x))f(\sigma(x))$ for x in D .*

PROOF. That the conditions are necessary for the existence of a contractive projection onto X is shown by Proposition 1.2 and Proposition 1.3. Proposition 2.2 shows these conditions are sufficient. We need only prove the statement about the one-to-one correspondence. By Proposition 1.2, for each contractive projection P , $c(x) = \frac{1}{2}(P(1_K)(x)+1)$ has the desired properties. For each c with these properties, it is shown in the proof of Proposition 2.2 that $P_c=R^{-1}T_c$ is a contractive projection onto X . The

proof is complete if we show that $c(x) = \frac{1}{2}(P_c(1_K)(x) + 1)$ for all x in $S \cup D$. But for x in $S \cup D$, $P_c(1_K)(x) = R^{-1}T_c(1_K)(x) = T_c(1_K)(x)$; and $T_c(1_K)(x) = c(x) - (1 - c(x)) = 2c(x) - 1$ for x in D and $T_c(1_K)(x) = 1 = 2c(x) - 1$ for x in S . Hence, we have that for x in $S \cup D$, $\frac{1}{2}(P_c(1_K) + 1) = \frac{1}{2}(2c(x) - 1 + 1) = c(x)$.

REMARKS. In [4] it is shown that if X is a separating subspace of $C(K)$ with no double extreme points, there can be at most one contractive projection onto X ; and furthermore there is a contractive projection onto X iff X is isometric by restriction to S , the closure of the single extreme points, to $C(S)$. This is a special case of Theorem 3.1.

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