HOMOLOGY TANGENT BUNDLES AND UNIVERSAL BUNDLES¹

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ABSTRACT. We find results about the evaluation map from the group of homeomorphisms of a closed manifold M and also about fibre bundles where M is the fibre. These facts follow from the observation that the homology tangent bundle is induced from a universal bundle pair.

1. Introduction. The object of this paper is to prove the following theorem:

THEOREM 12. *Let G be any group of homeomorphisms acting on a closed oriented topological manifold M and let $\omega: G \rightarrow M$ be the evaluation map at the base point $*$. Then $\chi(M)\omega^*: \tilde{H}^*(M; R) \rightarrow \tilde{H}^*(G; R)$ is trivial where R is any ring of coefficients with a unit and $\chi(M)$ is the Euler-Poincaré number of M .*

Note that the theorem applies to the coset mapping $\rho: G \rightarrow G/H$.

The proof of Theorem 12 makes use of the following observation: Let G be the path-connected component of the group of homeomorphisms of any topological manifold M and let H be the isotropy subgroup at $*$. Then we have the fibre bundle $M \rightarrow {}^iB_H \rightarrow B_G$.

THEOREM 5. *The inclusion i induces the homology tangent bundle from a "universal" fibre pair over B_H .*

We use a theorem of R. F. Brown to obtain Theorem 12 from Theorem 5. Along the way we obtain a topological version of a theorem of Borel [2].

THEOREM 9. *Let $M \rightarrow E \rightarrow {}^{\pi}B$ be any fibre bundle with an orientation preserving structural group, where M is a closed, orientable, topological manifold. Let p be a prime such that $\chi(M) \not\equiv 0 \pmod{p}$. Then $\pi^*: H^*(B; \mathbb{Z}_p) \rightarrow H^*(E; \mathbb{Z}_p)$ is injective.*

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2. Preliminaries. Let G be a topological group and let H be a closed subgroup. Let E be a space on which G operates on the right such that the projections $E \rightarrow E/G$ and $E \rightarrow E/H$ are principal fibre bundles with fibres G and H respectively. Also, we have the fibre bundle $\rho: E/H \rightarrow E/G$ with fibre G/H .

Let G operate on the left of a space F . Then, as usual, $E \times_G F$ will denote the quotient space of $E \times F$ under the equivalence relation $(e, x) \sim (eg, g^{-1}x)$. The equivalence class of (e, x) will be denoted by $\langle e, x \rangle$. Thus $\langle e, x \rangle$ is a point of $E \times_G F$ and $\langle e, x \rangle = \langle eg, g^{-1}x \rangle$. Similarly, we shall denote points of E/H by $\langle e \rangle_H$ where $e \in E$ and points of E/G will be denoted $\langle e \rangle_G$.

The map $p: E \times_G F \rightarrow E/G$ given by $p(\langle e, x \rangle) = \langle e \rangle_G$ is the projection of a fibre bundle with fibre F and group G .

Also, we need the concept of bundles induced by a map. Let $E \rightarrow {}^p B$ be a bundle and $X \rightarrow {}^f B$ be a map. The induced bundle $f^*(X) \rightarrow {}^{p(f)} B$ is a bundle with the same fibre and $f^*(X)$ is the subspace of $E \times X$ given by points of the form (e, x) such that $p(e) = f(x)$. We shall denote points in $f^*(X)$ by (e, x) . Then the projection map $p(f)$ is given by $(e, x) \mapsto x$.

We wish to consider the square of the bundle $E/H \rightarrow {}^\rho E/G$. This is the bundle $G/H \rightarrow \rho^*(E/H) \rightarrow E/H$. The points of $\rho^*(E/H)$ have the form $(\langle e \rangle_H, \langle e' \rangle_H)$ where $\langle e \rangle_G = \langle e' \rangle_G$. Note there is a unique $g \in G$ such that $e' \cdot g = e$.

There is another bundle with fibre G/H over E/H . This is $G/H \rightarrow E \times_H G/H \rightarrow E/H$. These two bundles are equivalent. We record some standard facts below for the reader's convenience:

LEMMA 1. *The two bundles $\rho^*(E/H) \rightarrow E/H$ and $E \times_H G/H \rightarrow E/H$ are equivalent. The bundle equivalence is given by the map $\rho^*(E/H) \xrightarrow{\alpha} E \times_H G/H$ which sends $(\langle e \rangle_H, \langle e' \rangle_H) \mapsto \langle e, g^{-1}H \rangle$ where g is defined by the equation $e' \cdot g = e$.*

LEMMA 2. *There exists a cross-section $s: E/H \rightarrow \rho^*(E/H)$ to the fibration $\rho^*(E/H) \rightarrow E/H$ given by $\langle e \rangle_H \mapsto (\langle e \rangle_H, \langle e \rangle_H)$. The fibration restricts over the fibre G/H of $E/H \rightarrow {}^\rho E/G$ to projection on the second factor $G/H \times G/H \rightarrow G/H$. Then the cross-section s restricted to the fibre G/H is the diagonal $G/H \xrightarrow{\Delta} G/H \times G/H$. We may also regard s as the cross-section $E/H \rightarrow E \times_H G/H$ given by $\langle e \rangle_H \mapsto \langle e, H \rangle$.*

LEMMA 3. *There is a commutative diagram of fibre bundles*

$$\begin{array}{ccccc}
 G/H \times G/H & \rightarrow & \rho^*(E/H) & \rightarrow & E/H \\
 \Delta \uparrow \downarrow p_2 & & s \uparrow \downarrow & & \downarrow p \\
 G/H & \xrightarrow{i} & E/H & \xrightarrow{\rho} & E/G
 \end{array}$$

LEMMA 4. *There is a bundle equivalence*

$$\begin{array}{ccc} E/H & \xrightarrow{\mu} & E \times_G G/H \\ \downarrow \rho & & \downarrow p \\ E/G & \xrightarrow{1} & E/G \end{array}$$

where $\mu(\langle e \rangle_H) = \langle e, H \rangle$.

3. The homology tangent bundle. In this section we shall always assume that G/H is a topological manifold without boundary, which we shall also denote by M . Also, G will be a group of homeomorphisms of M onto itself, endowed with the compact open topology. Then H will be a space of homeomorphisms of M which leave the base point $* \in M$ fixed, i.e. the isotropy subgroup. We shall always take E to be contractible. Thus we have $B_G = E/G$ and $B_H = E/H$.

Now observe that H may be regarded as a group of homeomorphisms of $M - *$. Consider the universal principal bundle $H \rightarrow E \rightarrow B_H$. By regarding H as operating on M , we obtain the associated bundle $M \rightarrow E \times_H M \rightarrow B_H$. By regarding H as operating on $M - *$ we obtain the associated bundle $(M - *) \rightarrow E \times_H (M - *) \rightarrow B_H$.

Observe that $E \times_H (M - *)$ is contained as a subspace in $E \times_H M$. In fact, using the cross-section s of Lemma 2, we have $E \times_H (M - *) = (E \times_H M) - s(B_H)$. The pair $(E \times_H M, E \times_H (M - *))$ is a fibre-bundle pair in the sense of Fadell [6]. See also [10, p. 256]. The fibre is $(M, M - *)$ and the base is B_H .

By Lemma 1, we may view the fibre pair from a different point. Recall the fibre bundle $M \xrightarrow{i} B_H \xrightarrow{\rho} B_G$. Then $E \times_H M$ is just $\rho^*(B_H)$, and $E \times_H (M - *)$ is $\rho^*(B_H) - s(B_H)$. Thus it is easy to see, using Lemma 3, that the fibre bundle $M \times M \xrightarrow{p_2} M$ given by projection on the second factor is induced by $i: M \rightarrow B_H$ from the bundle $\rho^*(B_H) \rightarrow B_H$. Also, the bundle $M \times M \xrightarrow{\Delta} M$ given by projection on the second factor is induced by $i: M \rightarrow B_H$ from the bundle $\rho^*(B_H) - s(B_H) \rightarrow B_H$. We formalize this in the following theorem.

THEOREM 5. *The bundle pair $(M \times M, M \times M - \Delta) \rightarrow {}^{p_2}M$ with fibre $(M, M - *)$ is induced from the bundle pair $(\rho^*(B_H), \rho^*(B_H) - s(B_H)) \rightarrow B_H$ by the inclusion map $i: M \rightarrow B_H$.*

Now let (E', E'_0) be a fibre pair with fibre (F, F_0) and base space B . Then $\pi_1(B, *)$ operates on $H_*(F, F_0; Z)$. The fibre pair (E', E'_0) is said to be *orientable* if $\pi_1(B, *)$ operates trivially on $H_*(F, F_0; Z)$. If $\pi_1(B, *)$ operates trivially on $H_*(F, F_0; Z_2)$, the fibre pair is called Z_2 -orientable.

LEMMA 6. (a) *The homology tangent bundle*

$$(M, M - *) \rightarrow (M \times M, M \times M - \Delta) \xrightarrow{P_2} M$$

is Z_2 -orientable. If M is orientable, then the homology tangent bundle is orientable.

(b) *Let M be orientable and let G be a group of homeomorphisms of M which preserve orientation. Then the fibre pair*

$$(M, M - *) \rightarrow (\rho^*(B_H), \rho^*(B_H) - s(B_H)) \rightarrow B_H$$

is orientable.

(c) *For any M and G , the above fibre pair is Z_2 -orientable.*

PROOF. (a) was proved by Fadell in [6].

(b) Let $\alpha \in \pi_1(B_H, *)$. We shall show that α acts trivially on $H_n(M, M - *)$. There exists an isotopy $f_t: M \rightarrow B_H$ such that:

(1) each f_t is a homeomorphism of M onto a fibre of $B_H \rightarrow^p B_G$.

(2) f_0 is the identity map of M regarded as the fibre over $p(*)$,

(3) the trace of the isotopy (i.e. the path $\sigma: t \mapsto f_t(*)$) is a closed path representing α .

Now we define an isotopy $g_t: M \rightarrow \rho^*B_H$ by setting $g_t(m) = (f_t(m), f_t(*)) \in \rho^*B_H$. Note that g_t is an isotopy of the pairs $(M, M - *) \rightarrow (\rho^*(B_H), \rho^*(B_H) - s(B_H))$.

Now g_0 is the identity on the fibre $(M, M - *)$ over $* \in B_H$. On the other hand $g_1(m) = (f_1(m), *) \in (M, M - *)$. Since f_1 preserves orientation on M by hypotheses, we see that $g_1: (M, M - *) \rightarrow (M, M - *)$ must induce the identity homomorphism on $H_n(M, M - *)$. Thus α acts trivially on $H_n(M, M - *)$, which was to be shown.

(c) is obvious since $H_n(M, M - *; Z_2) \cong Z_2$.

Let $(E', E'_0) \rightarrow B$ be an orientable fibre bundle pair with fibre $(M, M - *)$. Since $H^*(M, M - *) = H^*(R^m, R^m - 0)$, there exists a "Thom isomorphism" $\phi: H^i(B) \cong H^{i+n}(E', E'_0)$ which is natural with respect to mappings of fibre bundle pairs. Then $\phi(1) \in H^n(E', E'_0; R)$, where R is a ring of coefficients, is the "Thom class". Characteristic classes (Fadell [6]) are defined from the Thom class in the obvious way. For example, $\phi^{-1}(\phi(1) \cup \phi(1))$ is the Euler class.

The characteristic classes of M are defined to be the characteristic classes of the homology tangent bundle $(M, M \times M - \Delta)$.

COROLLARY 7. *If $k \in H^i(M; R)$ is a characteristic class of M , then k is in the image of $i^*: H^i(B_H; R) \rightarrow H^i(M; R)$.*

Now we assume that M is a closed orientable n -manifold. Then R. F. Brown [4] says the Euler class of M is equal to $\chi(M)\mu$ where $\mu \in H^n(M; R)$

is the fundamental class of the manifold and $\chi(M)$ is the Euler-Poincaré number of M . If $\varepsilon \in H^n(B_H; R)$ is the Euler class for the fibre pair $(\rho^*(B_H), \rho^*(B_H) - s(B_H)) \rightarrow B_H$, then by Theorem 5 we have $i^*(\varepsilon) = \chi(M)\mu$.

COROLLARY 8. $i^*(\varepsilon) = \chi(M)\mu$.

4. Proof of Theorem 9.

THEOREM 9. Let $M \rightarrow E \rightarrow {}^p B$ be a fibre bundle with orientation preserving structural group and let M be a closed, orientable topological manifold. If $\chi(M) \not\equiv 0 \pmod{p}$, then p^* is injective. In addition, the theorem is still true when $p=2$ with no orientability condition on M or on the fibre bundle.

PROOF. We know that the Euler class of M is $\chi(M)\mu \in H^n(M; \mathbb{Z})$, where μ is a generator, by a theorem of R. F. Brown [4]. Let $\chi(M)\mu$ also denote the element in $\mathbb{Z}_p$ cohomology. If $\chi(M) \not\equiv 0 \pmod{p}$, then

$$\chi(M)\mu \neq 0 \in H^n(M; \mathbb{Z}_p).$$

Consider the fibration $M \rightarrow {}^i B_H \rightarrow {}^p B_G$. By Corollary 8, $\chi(M)\mu$ is in the image of $i^*: H^n(B_H; \mathbb{Z}_p) \rightarrow H^n(M; \mathbb{Z}_p)$. (Note that in order to apply Corollary 8 we need that Lemma 6 be valid. But Lemma 6(b) requires that G be orientation preserving.) For $\mathbb{Z}_2$ coefficients we apply (c). By naturality, $\chi(M)\mu$ is in the image of i^* for the fibration $M \rightarrow {}^i E \rightarrow {}^p B$.

Lemma 3.1 of [2] (or see Theorem 14.5 of [3]) says the following: Suppose $F \rightarrow {}^i E \rightarrow {}^p B$ is a fibration with an n -dimensional fibre F such that $\pi_1(B)$ acts trivially on $H^n(F; \mathbb{Z}_p)$. If $i^*: H^n(E; \mathbb{Z}_p) \rightarrow H^n(F; \mathbb{Z}_p)$ is nonzero, then p^* is injective.

In the case at hand, $\pi_1(B)$ operates trivially on $H^n(M; \mathbb{Z}_p)$ since $\pi_1(B)$ operates trivially on the image of i^* (which is onto $H^n(M; \mathbb{Z}_p)$).

REMARK. In a forthcoming paper, we shall remove the hypothesis that M is orientable.

5. Actions and characteristic classes. In this section we shall prove that a characteristic class of a topological manifold M is "inert" under the action $\hat{\omega}: G \times M \rightarrow M$. We say $k \in H^*(M; R)$ is *inert* under $\hat{\omega}$ if $\hat{\omega}^*(k) = 1 \times k$, where R is any ring of coefficients. One consequence of the inertness of characteristic classes is that $\chi(M)\omega^*$ is trivial where $\omega: G \rightarrow M$ is evaluation at a base point $*$.

We begin by studying the action $\hat{\omega}: G \times (G/H) \rightarrow G/H$ given by $\hat{\omega}(g, g'H) = gg'H$. This action fits inside the following commutative diagram.

$$\begin{array}{ccc}
 G \times (G/H) & \xrightarrow{\hat{\omega}} & G/H \\
 \downarrow i \times 1 & & \downarrow i \\
 E \times (G/H) & \xrightarrow{\phi} & B_H = E/H \\
 \downarrow p \times * & & \downarrow p \\
 B_G & \xrightarrow{1} & B_G
 \end{array}
 \quad (*)$$

where ϕ is defined by $\phi(e, gH) = \langle eg \rangle_H$ and $i: G \rightarrow E$ is given by $i(g) = g$ and $i: G/H \rightarrow E/H$ is given by $i(gH) = \langle g \rangle_H$. Observe that ϕ is well defined and continuous and that $(*)$ in fact commutes.

Since E is contractible, the commutativity of $(*)$ tells us that $i \circ \hat{\omega}$ is homotopic to $G \times (G/H) \rightarrow^{\text{proj}} G/H \rightarrow^i B_H$. This proves the following theorem.

THEOREM 10. *Any k which is in the image of $i^*: H^*(B_H; R) \rightarrow H^*(G/H; R)$ is inert under the action $\hat{\omega}$. In particular, characteristic classes of topological manifolds are inert under $\hat{\omega}$.*

REMARK. The inertness of characteristic classes of differentiable manifolds under differentiable actions will be shown in [9] (i.e. Stiefel-Whitney classes and Pontrjagin class are inert under differentiable actions).

The usefulness of the concept of inertness comes from the following lemma. Let $\omega: G \rightarrow M$ be given by $\omega = \hat{\omega}(\cdot, *)$ for base point $* \in M$.

LEMMA 11. *Suppose that u and v are positive dimensional cohomology elements of a space M , and suppose that v is inert under some action $\hat{\omega}$. Then $u \cup v = 0$ implies $\omega^*(u) \times v = 0$.*

PROOF. We have

$$\begin{aligned}
 0 &= \hat{\omega}^*(u \cup v) = \hat{\omega}^*(u) \cup \hat{\omega}^*(v) \\
 &= (\omega^*(u) \times 1 + \sum a_i \times b_i + \sum c_i * d_i) \cup 1 \times v \\
 &= \omega^*(u) \times k + \sum a_i \times (b_i \cup k) + \sum (c_i * d_i) \cup (1 \times v).
 \end{aligned}$$

Here the a_i and $c_i \in H^*(M; R)$ are positive dimensional and b_i and $d_i \in H^*(G; R)$ and $\times$ is the cohomology cross product and $*$ is the torsion product coming from the Künneth formula. It is easy to see that no term in the expansion has the right dimensions to cancel $\omega^*(u) \times k$. Hence $\omega^*(u) \times k = 0$.

The above lemma, with the aid of Corollary 8 gives us the main result.

THEOREM 12. *Let M be a closed orientable topological manifold and let G be a group of homeomorphisms acting on M by the action $\hat{\omega}: G \times M \rightarrow M$. Let $\omega: G \rightarrow M$ be the evaluation map at the base point $*$. Then $\chi(M)\omega^*: \tilde{H}^*(M; R) \rightarrow \tilde{H}^*(G; R)$ is trivial where R is any coefficient ring with unit.*

PROOF. By Corollary 8, $i^*(\epsilon) = \chi(M)\mu$. So $\chi(M)\mu$ is inert by Theorem 10. (We let $\chi(M)\mu$ also stand for the image of $\chi(M)\mu$ in cohomology with coefficients in R .) Since μ is the top dimensional class, $\mu \cup v = 0$ for any $v \in \tilde{H}^*(M; R)$. Thus, by Lemma 11,

$$0 = \omega^*(v) \times \chi(M)\mu = \chi(M)\omega^*(v) \times \mu.$$

So $\chi(M)\omega^*(v)$ must equal zero, thus proving the theorem.

REMARK. If $R = \mathbb{Z}_2$, the orientability requirement may be dropped.

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