PROJECTIVE REPRESENTATIONS OF ABELIAN GROUPS

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ABSTRACT. Let ω be a factor system for the locally compact abelian group G . Then we show that the finite-dimensional unitary irreducible projective representations of G , having factor system ω , possess a common dimension $d(\omega)$. Using a characterisation of $d(\omega)$ as the index in G of a maximal subgroup on which ω is symmetric we derive a formula for $d(\omega)$ in the case that G is discrete and finitely generated.

1. Introduction. This paper developed out of a successful attempt to prove the following: if D_1 and D_2 are finite-dimensional irreducible unitary projective representations of a locally compact abelian group G , possessing the same factor system ω , then D_1 and D_2 have the same dimension $d(\omega)$. Theorem 1 contains this result and states a fortiori that D_1 and D_2 can differ essentially only by a linear character of G .

Although in the general case it may not be possible to produce a formula for $d(\omega)$ in terms of ω , some progress in this direction has been made in Theorem 3 with the identification of $d(\omega)$ as the index in G of a maximal subgroup on which ω is symmetric. This is not just an incidental result for it plays a role in establishing a procedure for the evaluation of $d(\omega)$ in the special case that G is discrete and finitely generated.

For the definitions and results appropriate to topological groups and their representations we refer the reader to Mackey [6] and [7], and in particular to Lemma 8.1 of [7] which is the key to the proof of Theorem 1. Although this lemma is only proved in the case of ordinary representations, to avoid duplication, we merely comment that most results about ordinary representations can be interpreted without essential alteration in terms of projective representations. We shall deal exclusively with finite-dimensional unitary projective representations, and shall use the term ω -representation to indicate the appropriate factor system and the term ω -rep in irreducible cases. Also we assume that all groups are locally compact, though we do not restrict attention to abelian groups until Theorem 1.

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2. The main theorem. Let D be an ω -representation of G , then according to Mackey [6], $\bar{D}$, the adjoint of D , is an $\bar{\omega}$ -representation.

Concretely, if we consider D as a matrix representation, then the matrices which correspond to $\bar{D}$ are the complex conjugates of those which correspond to D . Then we have the following result.

LEMMA 1. *Let D_1 and D_2 be ω -representations of the group G . If D_2 is irreducible then D_1 contains D_2 as a direct summand if and only if the Kronecker product $D_1 \otimes \bar{D}_2$ contains the trivial representation of G .*

PROOF. To establish this result we form the lifted representations $D_1 \nearrow G^\omega$ and $D_2 \nearrow G^\omega$, see Backhouse and Bradley [2], and apply directly Lemma 8.1 of Mackey [6]. Alternatively we note that the proof of Lemma 8.1 of [6] is valid word for word in the case of projective representations.

As an immediate application we have the following result, being a generalisation of a well-known property of ordinary representations.

LEMMA 2. *Let D_i be an ω_i -rep, having dimension d_i , of the group G , for $i=1, 2, 3$, where $\omega_3=\omega_1\omega_2$.*

If D_3 is contained in $D_1 \otimes D_2$ as a direct summand, then $\bar{D}_2$ is contained in $D_1 \otimes \bar{D}_3$ and $\bar{D}_1$ is contained in $D_2 \otimes \bar{D}_3$. Furthermore, $d_3 \leq d_1 d_2$, $d_2 \leq d_1 d_3$ and $d_1 \leq d_2 d_3$.

PROOF. The hypothesis, together with Lemma 1, implies that $D_1 \otimes D_2 \otimes \bar{D}_3$ contains the trivial representation of G . Then noting that $D_2 = \bar{D}_2$ and with the commutativity and associativity of Kronecker products, the converse of Lemma 1 implies that $D_1 \otimes \bar{D}_3$, $D_2 \otimes \bar{D}_3$, contain $\bar{D}_2$ and $\bar{D}_1$, respectively, as required. Finally, the dimensionality constraints follow because the dimension of a subrepresentation cannot exceed that of its containing representation.

We now prove the main theorem.

THEOREM 1. *Let D_1 and D_2 be two ω -reps of the abelian group G . Then there exist a unitary transformation U and a linear character χ where*

$$(2.1) \quad U^{-1}D_1(g)U = \chi(g)D_2(g),$$

for all $g \in G$.

PROOF. Let D_i be of dimension d_i for $i=1, 2$. Since $\bar{D}_2$ is an $\bar{\omega}$ -rep of G it follows that $D_1 \otimes \bar{D}_2$ is an ordinary representation of G , and since G is abelian, contains at least one linear character χ . Now Lemma 2 implies that $d_1 \leq d_2$ and $d_2 \leq d_1$, and hence that $d_1 = d_2$.

Lemma 2 tells us further that $\bar{D}_1$ is contained in $\bar{D}_2 \otimes \bar{\chi}$, which being of equal dimension must be unitarily equivalent, and hence (4.1) follows.

3. Maximal symmetric subgroups. Since the problem of describing the factor systems of an arbitrary abelian group is as yet unsolved, see, however, the work of Kleppner [4], it is unreasonable to expect a complete analysis of the dimension function $d(\omega)$. However, the knowledge of the existence of an ω -rep D of dimension $d(\omega)$ does impose certain conditions on ω . For by definition

$$(3.1) \quad D(g_1)D(g_2) = \omega(g_1, g_2)D(g_1g_2),$$

$$(3.2) \quad = \omega(g_1, g_2)D(g_2g_1),$$

$$(3.3) \quad = c(g_1, g_2)D(g_2)D(g_1),$$

where $c(g_1, g_2) = \omega(g_1, g_2)/\omega(g_2, g_1)$, for all $g_1, g_2 \in G$. Taking the determinant of both sides of (3.3) it follows that $c(g_1, g_2)$ is a $d(\omega)$ th root of unity for all $g_1, g_2 \in G$. Further it has been shown by Kleppner [4] that c is a bicharacter of G , that is, it is linear in each variable separately. We can at once deduce the following.

THEOREM 2. *If G is a divisible abelian group then $d(\omega)$ (if it is finite) is unity for every factor system ω .*

PROOF. Since G is divisible the bicharacter takes the value one throughout G . But then ω is symmetric on G , and so it follows from Theorem 2.1 of [3] that ω is equivalent to the trivial factor system. Hence all ω -reps of G are projectively equivalent to linear characters, which are of dimension one.

For example Theorem 2 holds if G is a vector group. Hence the only ω -reps of R^n are the linear characters.

In general ω is not symmetric throughout G . However, since ω is symmetric on the identity subgroup, it follows from Zorn's lemma that there exists at least one maximal subgroup of G on which ω is symmetric. It turns out that such a subgroup is relatively large, and indeed closely linked to the number $d(\omega)$.

THEOREM 3. *If M is a maximal subgroup of the abelian group G on which ω is symmetric, then M has index $d(\omega)$ in G .*

PROOF. We first claim that if $a_1 \notin M$ then M is of index r_1 , where $2 \leq r_1 \leq d(\omega)$, in the group $M(a_1)$ obtained by adjoining a_1 to M . For, if $t = a_1^{d(\omega)}$ then $c(t, g) = c(a_1, g)^{d(\omega)} = 1$ for all $g \in M$. The bilinearity of c implies that ω is symmetric on the group obtained by adjoining t to M . The maximality of M then implies that $t \in M$, and hence that M is of index r_1 in $M(a_1)$, where r_1 is a divisor of $d(\omega)$ larger than unity.

If there exists $a_2 \notin M(a_1)$, then it is evident that M is of index r_2 , where $4 \leq r_2 \leq d(\omega)^2$, in the group $M(a_1, a_2)$ obtained by adjoining a_2 to $M(a_1)$.

Continuing in this manner we decide to stop either if we find M to be of index $\leq d(\omega)$ in G or if we find a subgroup H of G in which M has index $> d(\omega)$.

Since ω is symmetric on M it is possible to write it in the form

$$(3.4) \quad \omega(g, g') = a(g)a(g')/a(g, g'),$$

for all $g, g' \in M$, for some T -valued function a . It follows that an ω -rep of M is of the form $g \rightarrow a(g)b(g)$, where b is a linear character of M . Since M is of finite index in G or H we can apply the little ω -group method, see Backhouse and Bradley [2], to this ω -rep of M to yield an ω -rep of G or H . As our initial representation is 1-dimensional its little ω -group consists of all those $h \in G$ or H which satisfy

$$(3.5) \quad \omega(g, h)\omega(h^{-1}, gh) = \omega(h^{-1}, h),$$

for all $g \in M$. Rewriting (3.5) we obtain

$$(3.6) \quad (\omega(g, h)/\omega(h, g))\omega(h^{-1}, hg)\omega(h, g) = \omega(h^{-1}, h),$$

for all $g \in M$, and so

$$(3.7) \quad (\omega(g, h)/\omega(h, g))\omega(e, g)\omega(h^{-1}, h) = \omega(h^{-1}, h),$$

using (2.5) of [2], and then $\omega(g, h) = \omega(h, g)$, for all $g \in M$, using (2.3) of [2]. The maximality of M now implies that $h \in M$. It follows that the induced ω -representation $ab \uparrow G$ or H is irreducible. In the latter case $ab \uparrow H$ is of dimension $> d(\omega)$, which is a contradiction, since by restriction H has ω -reps of dimension $\leq d(\omega)$. Otherwise $ab \uparrow G$ gives an ω -rep of G , and since this is of dimension $d(\omega)$ it shows that M is of index $d(\omega)$ in G .

COROLLARY. *Every ω -rep of the abelian group G is projectively equivalent to an induced linear character of some subgroup.*

4. Finitely generated abelian groups. Unfortunately the characterisation of $d(\omega)$ given in Theorem 3 does not lead directly to a formula for $d(\omega)$. Even in cases where ω is easily described, for example if G is finitely generated, it can be very difficult to find any maximal subgroups on which ω is symmetric. Indeed the following analysis of $d(\omega)$ for a finitely generated group shows just how complicated and unwieldy a maximal symmetric subgroup can be.

We first note that a finitely generated abelian group G is a factor group of an appropriate free abelian group T_n on a finite number of generators n . Furthermore since a factor system ω and an associated ω -rep of G can be lifted to a factor system and ω -rep of T_n it is sufficient in evaluating $d(\omega)$ to concentrate on T_n .

If ω is a factor system of T_n , then without loss of generality it can be taken in the bilinear form as given in Theorem 4.1 of [2]. Thus, using a vector notation for the elements of T_n ,

$$(4.1) \quad \omega(t, t') = \exp(-2\pi i t' G t'),$$

for all $t, t' \in T_n$, where G is an $n \times n$ lower triangular or skew matrix. It is easy to show that $d(\omega)$ exists only if G has rational entries. Suppose we change the basis of T_n , then with respect to this new basis the matrix determining ω is $U^t G U$, where U , representing a basis transformation, is a unimodular integer matrix. The idea is to choose U in such a manner that as many as possible of the entries of $U^t G U$ are zero. In this connection we shall presently see the application of the following result.

LEMMA 3. *Let C be an integer skew-symmetric $n \times n$ matrix. Then there exists a unimodular integer matrix U such that $U^t C U$ is zero except for a certain number r ($2r \leq n$) of 2×2 integer skew-symmetric matrices E_i ($i=1, 2, \dots, r$) placed along the diagonal. Furthermore it is possible to choose the matrices E_i in such a way that the nonzero entries of E_i divide the nonzero entries of E_{i+1} for $1 \leq i \leq r-1$.*

An inductive proof may be based on the case $n=3$ for which it is possible to explicitly calculate U . This result is due to Frobenius, and arose in his work on theta functions (see also Lang [5, p. 380]). The chief virtue of Lemma 3 is that we can explicitly determine the nonzero entries in $U^t C U$ without having to work out U itself. This statement is based on the following result of H. J. S. Smith (see for example Aitken and Turnbull [1]).

LEMMA 4. *If $B=PAQ$, where P, A and Q , are $n \times n$ integer matrices, then the highest common factor of the $r \times r$ minors of A divides the highest common factor of the $r \times r$ minors of B , for every $r \leq n$.*

COROLLARY. *If $S=U^t C U$, where U, C , are $n \times n$ integer matrices with U unimodular, then the highest common factors of the $r \times r$ minors of C and S are equal, for every $r \leq n$.*

We can now write down a procedure for evaluating $d(\omega)$.

1. Let the matrix G which represents ω be in lower triangular form with its entries $G_{ij} = t_{ij}/N_{ij}$, for $i > j$, as reduced fractions. Now add to G the symmetric matrix whose (i, j) entry is $-\frac{1}{2}G_{ij}$, for $1 \leq i, j \leq n$, to form the skew-symmetric matrix A .

2. If N is the least common multiple of the N_{ij} 's, then $A = (1/2N)C$, where C is a skew-symmetric integer matrix.

3. According to Lemma 3 there exists a unimodular integer matrix U such that

$$(4.2) \quad U^t C U = E_1 \oplus E_2 \oplus \cdots \oplus E_r \oplus O_{n-2r},$$

where

$$E_i = \begin{pmatrix} 0 & -a_i \\ a_i & 0 \end{pmatrix}$$

is a nonzero 2×2 integer matrix for $i=1, 2, \dots, r$, r is the rank of A , and $a_i | a_{i+1}$ for $i=1, 2, \dots, r-1$.

4. Since a change of basis can have no effect on $d(\omega)$ we deduce from Theorem 3 that

$$(4.3) \quad d(\omega) = \prod_{i=1}^r d_i,$$

where d_i is the dimension corresponding to the factor system of T_2 obtained from the matrix $(1/2N)E_i$. It follows from §5 of [2] that d_i is $N/(N, a_i)$.

5. If h_i is the highest common factor of the $(2i-1) \times (2i-1)$ minors of C , then for $i=1, 2, \dots, r$ we have

$$(4.4) \quad h_i = a_1^2 a_2^2 \cdots a_{i-1}^2 a_i,$$

and if k_i is the highest common factor of the $2i \times 2i$ minors of C , then for $i=1, 2, \dots, r$ we have

$$(4.5) \quad k_i = a_1^2 a_2^2 \cdots a_i^2.$$

If we put $k_0=1$, then

$$(4.6) \quad a_i = h_i/k_{i-1},$$

$$(4.7) \quad = (k_i/k_{i-1})^{1/2},$$

for $i=1, 2, \dots, r$. Now we can write

$$(4.8) \quad d(\omega) = N^r / \prod_{i=1}^r (N, a_i),$$

where a_i is given through (4.6) or (4.7) in terms of the integers NG_{ij} , $1 \leq i, j \leq n$.

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