ON THE CONGRUENCE LATTICES OF UNARY ALGEBRAS

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ABSTRACT. Characterizations of those unary algebras whose congruence lattices are semimodular or atomic are obtained. Combining these results gives necessary and sufficient conditions for a unary algebra to have a geometric congruence lattice.

A unary algebra, $\mathfrak{A} = \langle A; f \rangle$, is a set A together with a function f from A into A . A *congruence relation* θ of $\mathfrak{A}$ is an equivalence relation on A such that if $x \equiv y(\theta)$ then $f(x) \equiv f(y)(\theta)$. The congruence lattice of $\mathfrak{A}$, denoted $\Theta(\mathfrak{A})$, is the collection of all congruence relations of $\mathfrak{A}$ partially ordered in the usual manner [2, p. 50]. If L is a lattice and $a, b \in L$, then b is said to cover a if $a < b$ and $\{x | a < x < b\} = \emptyset$. This relation will be denoted $a < b$. L is *semimodular* if for any a, b and $c \in L$ such that $c < a$ and $c < b$ then $a < a \vee b$ and $b < a \vee b$. Theorem 1 gives a characterization of those unary algebras whose congruence lattices are semimodular. An *atom* of a lattice with 0 is an element that covers 0. A lattice is *atomic* if every element is a join of atoms. Theorem 2 gives a characterization of those unary algebras whose congruence lattices are atomic.

Let $\mathfrak{A} = \langle A; f \rangle$ be a unary algebra. For any $x \in A$ and any positive integer n let $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$. A *component* of $\mathfrak{A}$ is a subset of A such that x, y are in the same component if and only if there exist $m, n \in N$ such that $f^m(x) = f^n(y)$. Index the components of $\mathfrak{A}$ by an index set I so that $A = \bigcup_{i \in I} A_i$. For each component A_i , choose a fixed subset C_i , called a *cycle*, as follows: If there exists $n \in N$ and $x \in A_i$ such that $f^n(x) = x$, let $C_i = \{f^k(x) | k \in N\}$. Such a cycle is a unique finite subset of A_i . Denote such a finite cycle by $\{x_0, x_1, \dots, x_{m-1}\}$, with $f(x_i) = x_{i+1}$. If n is any integer let $x_n = x_i$ where $n \equiv i \pmod{m}$, $0 \leq i < m$. If no such x and n exist for A_i let x_0 be any fixed element of A_i , $C_i = \{f^m(x_0) | m \in N\}$. Denote this set by $\{x_0, x_1, x_2, \dots\}$ where $f(x_i) = x_{i+1}$. If C_i and C_j are any two cycles, $|C_i| = m_i$ and $|C_j| = m_j$, then C_i and C_j are said to have *relatively prime cardinalities* if both m_i and m_j are finite and $(m_i, m_j) = 1$ or if m_i is

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infinite and $m_j=1$. A general discussion of unary algebras can be found in [3] and [4].

If $B \subseteq A$, let $\theta(B)$ denote the smallest element of $\Theta(\mathfrak{A})$ for which B is contained in one congruence class. $\theta(\{x, y\})$ is usually denoted by $\theta(x, y)$. I_A and U_A will denote $\{\langle x, x \rangle \mid x \in A\}$ and $A \times A$ respectively. A congruence class is said to be nontrivial if it has more than one element. Finally, if L is any lattice, $a, b \in L$, then $[a, b] = \{x \mid x \in L \text{ and } a \leq x \leq b\}$.

LEMMA 1. *Let $\mathfrak{A} = \langle A; f \rangle$ and $\mathfrak{B} = \langle B; g \rangle$ and $\mathfrak{C} = \langle C; h \rangle$ be unary algebras such that $\mathfrak{B}$ is any homomorphic image of $\mathfrak{A}$ and $\mathfrak{C}$ is any subalgebra of $\mathfrak{A}$. Then $\Theta(\mathfrak{B})$ is isomorphic to a dual ideal of $\Theta(\mathfrak{A})$ and $\Theta(\mathfrak{C})$ is isomorphic to an ideal of $\Theta(\mathfrak{A})$.*

PROOF. The result for homomorphic images holds for any algebra [2, p. 61]. As for subalgebras, since $\mathfrak{A}$ is a unary algebra, C is the only nontrivial congruence class of $\theta(C)$. Hence the interval $[I_A, \theta(C)]$ in $\Theta(\mathfrak{A})$ is isomorphic to $\Theta(\mathfrak{C})$.

LEMMA 2. *Let C be any cycle of some unary algebra $\mathfrak{A} = \langle A; f \rangle$. Then $[I_A, \theta(C)]$ is semimodular.*

PROOF. If C is finite then $[I_A, \theta(C)]$ is isomorphic to the lattice of divisors of $|C|$ and is distributive and hence semimodular. If C is infinite and if $\theta \in [I_A, \theta(C)]$ then $\theta = \theta(x_m, x_{m+p})$ for fixed m and p . Also $\theta < \Psi$ if and only if $\Psi = \theta(x_{m+1}, x_{m+1+p})$ or $\Psi = \theta(x_m, x_{m+q})$ where p/q is prime. An examination of these possibilities shows semimodularity holds.

LEMMA 3. *Let $\mathfrak{A} = \langle A; f \rangle$ be a unary algebra. Then the atoms of $\Theta(\mathfrak{A})$ are of the following types:*

- (i) atoms of $[I_A, \theta(C_i)]$ where C_i is a finite cycle;
- (ii) $\theta(x, y)$ where $f(x) = f(y)$;
- (iii) $\theta(x, y)$ where $x \in C_i, y \in C_j, i \neq j, |C_i| = |C_j|$ and $|C_i|$ is finite.

PROOF. Let θ be an atom, $x \equiv y(\theta)$, $x \neq y$. Then if θ is not of type (ii) then $\theta = \theta(x, y) = \theta(f(x), f(y))$. This implies that $\{x, y\} \subseteq \{f^n(x) \mid n \geq 1\} \cup \{f^n(y) \mid n \geq 1\}$. Therefore x and y must be in finite cycles. If they are in the same cycle (i) must hold. If they are in different cycles (iii) must hold.

THEOREM 1. *Let $\mathfrak{A} = \langle A; f \rangle$ be a unary algebra. Then $\Theta(\mathfrak{A})$ is upper semimodular if and only if the cycles of $\mathfrak{A}$ have relatively prime cardinalities.*

PROOF. Suppose that the cycles of $\mathfrak{A}$ are of relatively prime cardinality and α, β , and τ are in $\Theta(\mathfrak{A})$, $\tau < \alpha$ and $\tau < \beta$. Let $\bar{\mathfrak{A}} = \mathfrak{A}/\tau$. It follows that the cycles of $\bar{\mathfrak{A}}$ are also of relatively prime cardinality. Also $\Theta(\bar{\mathfrak{A}}) \cong [\tau, U_A]$ in $\Theta(\mathfrak{A})$ by Lemma 1. Let $\bar{\alpha}$ and $\bar{\beta}$ be the images of α and β in

$\Theta(\overline{\mathfrak{U}})$. Thus $\bar{\alpha}$ and $\bar{\beta}$ are atoms of $\Theta(\overline{\mathfrak{U}})$. So any atom of type (iii) in Lemma 3 will have only one nontrivial congruence class and this will be of cardinality two.

It will be sufficient to show that $\bar{\beta} < \bar{\beta} \vee \bar{\alpha}$ and $\bar{\alpha} < \bar{\beta} \vee \bar{\alpha}$ in $\Theta(\overline{\mathfrak{U}})$. Note that if $\bar{\alpha}$ and $\bar{\beta}$ are such that their nontrivial congruence classes are disjoint then indeed $\bar{\beta} < \bar{\beta} \vee \bar{\alpha}$ and $\bar{\alpha} < \bar{\beta} \vee \bar{\alpha}$. Since $\bar{\beta}$ and $\bar{\alpha}$ are atoms and by Lemma 3 there are only three types of atoms, there are essentially only six different cases to consider:

$\bar{\alpha}$ and $\bar{\beta}$ both of type (i). Let $\bar{\alpha} \leq \theta(C_i)$ and $\bar{\beta} \leq \theta(C_j)$ for cycles C_i and C_j in $\overline{\mathfrak{U}}$. By Lemma 2 it can be assumed that $i \neq j$. Also

$$\bar{\alpha} \vee \bar{\beta} \in [I_A, \theta(C_i) \vee \theta(C_j)] \simeq [I_A, \theta(C_i)] \times [I_A, \theta(C_j)].$$

From Lemma 2 and the fact that the product of semimodular lattices is semimodular, semimodularity holds in this case.

$\bar{\alpha}$ of type (i) and $\bar{\beta}$ of type (ii). Let $\bar{\alpha} \leq \theta(C_i)$ and $\bar{\beta} = \theta(x, y)$ where $f(x) = f(y)$. $\bar{\beta}$ has only one nontrivial congruence class and this will be of cardinality two. By the remarks above concerning disjoint congruence classes it can be assumed that $x \in C_i$ and $y \notin C_i$. But then $\bar{\beta} \vee \bar{\alpha}$ has as its congruence classes those of $\bar{\alpha}$ with y adjoined to one of them. Clearly then $\bar{\beta} < \bar{\beta} \vee \bar{\alpha}$ and $\bar{\alpha} < \bar{\beta} \vee \bar{\alpha}$.

$\bar{\alpha}$ of type (i) and $\bar{\beta}$ of type (iii). Since the cycles of $\overline{\mathfrak{U}}$ are of relatively prime cardinalities, $\bar{\alpha}$ and $\bar{\beta}$ must have disjoint nontrivial congruence classes so the desired covering relations hold. The remaining three cases are handled similarly.

Conversely, it will be shown that the interval sublattice $[I_A, \theta(C_i \cup C_j)]$ is not semimodular if the cardinalities of C_i and C_j are not relatively prime. Let $|C_i| = m_i$ and $|C_j| = m_j$. If m_i and m_j are both finite let p be any prime that divides both of them. If only m_i is finite let p be any prime that divides m_i . If m_i and m_j are both infinite let p be any prime. Let $\{x_0, x_1, x_2, \dots\} \subseteq C_i$ and $\{y_0, y_1, y_2, \dots\} \subseteq C_j$ where $f(x_t) = x_{t+1}$ and $f(y_t) = y_{t+1}$. Let $\tau \in \Theta(\overline{\mathfrak{U}})$ be $\tau = \theta(x_0, y_0, x_p, y_p)$. Since p is a prime $\tau < \theta(x_0, y_0, x_1, y_1)$. Also $(\tau \wedge (\theta(x_0, x_1) \vee \theta(y_0, y_1))) = \theta(x_0, x_p) \vee \theta(y_0, y_p) < \tau$. Also $\theta(x_0, x_p) \vee \theta(y_0, y_p) < \theta(x_0, x_p) \vee \theta(y_0, y_1)$, since p is prime. This gives the following covering diagram:

This shows that semimodularity cannot hold in $[I_A, \theta(C_i \cup C_j)]$.

Note. If $\mathfrak{A}$ has only one component then $\Theta(\mathfrak{A})$ is semimodular. This gives, in the finite case, Theorem 3 of [6].

THEOREM 2. *Let $\mathfrak{A}$ be a unary algebra with components A_i and cycles C_i , $i \in I$. Then $\Theta(\mathfrak{A})$ is atomic if and only if $\mathfrak{A}$ satisfies the following conditions:*

- (i) $|C_i|$ is finite for all $i \in I$.
- (ii) $|C_i| = |C_j|$ for all $i, j \in I$.
- (iii) $|C_i|$ is a square free integer for all $i \in I$.
- (iv) For at most one $i \in I$ does $A_i \neq C_i$, in which case there exists $x_0 \in C_i$ such that for all $u \in A_i$, $u \notin C_i$, $f(u) = x_0$.

PROOF. Let $\mathfrak{A}$ be such that $\Theta(\mathfrak{A})$ is atomic. Then (i) must hold since if C_i is an infinite cycle, the interval $[I_A, \theta(C_i)]$ contains no atoms. If $|C_i| \neq |C_j|$ and $x_0 \in C_i$ and $y_0 \in C_j$ then, by Lemma 3, $\theta(x_0, y_0)$ is not an atom. But $\theta(x_0, y_0)$ is also join irreducible. Hence (ii) holds. Next, suppose contrary to (iii) that $|C_i| = p^2q$ where p is some prime number. Let x_0, x_q be in C_i . The element $\theta(x_0, x_q)$ is not an atom in $\Theta(\mathfrak{A})$ since it covers $\theta(x_0, x_{pq})$. But $\theta(x_0, x_q)$ is also join irreducible. This contradicts the fact that $\Theta(\mathfrak{A})$ is atomic. Now suppose that there exists $i \neq j$ such that $A_i \neq C_i$ and $A_j \neq C_j$. Let $u \in A_i \setminus C_i$ and $v \in A_j \setminus C_j$ be such that $f(u) = x_0 \in C_i$ and $f(v) = y_0 \in C_j$. $\theta(u, v)$ is not an atom since $\theta(u, v) > \theta(x_0, y_0) > I_A$. But again $\theta(u, v)$ is join irreducible. This shows that the first part of (iv) must hold. A similar argument applies to $u, v \in A_i \setminus C_i$, $f(u) \neq f(v)$, $f(u), f(v) \in C_i$. Finally, suppose there exists $u \in A_i \setminus C_i$, $f(u) \notin C_i$. Let $f^n(u) = x_0$, so $n > 1$. Consider $\theta(u, x_{-n})$; it is also nonatomic and join irreducible. Hence (iv) holds.

Conversely, suppose conditions (i)–(iv) hold. Let x_0 be as in condition (iv). Form $S = \{u \mid f(u) = x_0\}$. Then $[I_A, \theta(S)]$ is isomorphic to $\mathcal{E}(S)$, the lattice of all equivalence relations on S . Since $\mathcal{E}(S)$ is atomic, it follows that $[I_A, \theta(S)]$ is also. Let $\alpha \in \Theta(\mathfrak{A})$. Claim $\alpha = (\alpha \wedge \theta(\bigcup C_i)) \vee (\alpha \wedge \theta(S))$. For if $y_i \equiv u(\alpha)$ where $y_i \in C_j$ and $u \notin \bigcup C_i$, then $y_{i+1} \equiv x_0(\alpha)$ and hence $y_i \equiv x_{-1}(\alpha)$. But then $y_i \equiv x_{-1}(\alpha \wedge \theta(\bigcup C_i))$ and $y_i \equiv x_{-1} \equiv u(\alpha)$ so $x_{-1} \equiv u(\alpha \wedge \theta(S))$ so that $u \equiv y_i(\alpha \wedge \theta(\bigcup C_i)) \vee (\alpha \wedge \theta(S))$. This verifies the claim. Hence all that remains is to show that the interval $[I_A, \theta(\bigcup C_i)]$ is atomic. Conditions (i) and (iii) imply that $[I_A, \theta(C_i)]$ is atomic and hence $[I_A, \bigvee_{i \in I} \theta(C_i)]$ is also atomic. Condition (ii) guarantees that any element of the form $\theta(x, y)$ is an atom for $x \in C_k$ and $y \in C_j$, $k \neq j$. But any element in $[I_A, \theta(\bigcup_{i \in I} C_i)]$ is the join of elements $\theta(x, y)$ where either $x, y \in C_j$ or $x \in C_j$ and $y \in C_k$. But these are either atoms or joins of atoms so the proof is complete.

DEFINITION. A *geometric lattice* is a finite dimensional lattice that is both atomic and semimodular.

THEOREM 3. *A unary algebra $\mathfrak{A} = \langle A; f \rangle$ has a congruence lattice which is a geometric lattice if and only if A is finite and*

(1) *$\mathfrak{A}$ has exactly one cycle C , $|C|$ is square free and for all $x, y \in A \setminus C$, $f(x) = f(y) \in C$.*

(2) *Every cycle of $\mathfrak{A}$ is of cardinality 1, and if C is the union of all these cycles, then for all $x, y \in A \setminus C$, $f(x) = f(y) \in C$.*

PROOF. Combine the conditions of Theorems 1 and 2.

Note. If the conditions of Theorem 3 hold then it is easily verified that $\Theta(\mathfrak{A}) \simeq [I_A, \theta(C)] \times [I_A, \theta(T)]$ where $T = \{x \mid f(x) = f(y), x \neq y\}$.

COROLLARY. *Let L be a geometric lattice. Then L is isomorphic to the congruence lattice of some unary algebra if and only if $L \simeq 2^p \times \mathcal{E}(S)$ or $L \simeq \mathcal{E}(S) \times \mathcal{E}(T)$, where p is a finite positive integer and S and T are finite sets.*

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