VALENTINE'S EXTENSIONS OF TIETZE'S THEOREM ON CONVEX SETS

NICK M. STAVRAKAS AND R. E. JAMISON

ABSTRACT. New proofs are given for Valentine's extensions of Tietze's theorem on convex sets.

Valentine in [1] proved the following two extensions of Tietze's theorem on convex sets.

THEOREM 1. *Let S be a closed connected subset of R^d such that S has at most n points of local nonconvexity. Then S is an L_{n+1} set.*

THEOREM 2. *Let S be a closed connected subset of R^d such that the points of local nonconvexity of S can be decomposed into n convex sets. Then S is an L_{2n+1} set.*

The purpose of this paper is to give new proofs of these theorems. The proofs depend upon the following two lemmas.

LEMMA 1. *Let S be a subset of R^d . Let $x, y \in S$ and suppose l is an arc from x to y of minimal arc length in S such that each point of l is a point of local convexity of S , except possibly for x and y . Then $l = [xy]$, the closed line segment from x to y .*

LEMMA 2. *Let S be a closed connected subset of R^d , which has at least one point of local nonconvexity. Then given $x \in S$ there exists a point p of local nonconvexity of S such that $[xp] \subset S$.*

The proof of Lemma 1 is easy and is omitted. The proof of Lemma 2 is in Valentine [1]. We only prove Theorem 2, and then it will be clear how to prove Theorem 1 using the same method.

PROOF OF THEOREM 2. Let $\{C_1, \dots, C_n\}$ be the n convex sets into which the points of local nonconvexity can be decomposed. Now let $z \in S$. Define $S(z)$ as $\{t \mid t \in S \text{ and } [zt] \subset S\}$ and define $S(C_i)$ as $\bigcup_{z \in C_i} S(z)$. Clearly $S(C_i)$ is an L_3 set. Lemma 2 implies $S = \bigcup_{i=1}^n S(C_i)$. The fact that S is connected then implies that S is an L_{3n} set. Thus, given $x, y \in S$, there exists an arc from x to y in S of finite arc length, and hence there exists an

Received by the editors November 24, 1971.

AMS 1970 subject classifications. Primary 52A20; Secondary 52A15.

© American Mathematical Society 1972

arc from x to y of minimal arc length in S . Let I be such an arc. I is clearly simple. The minimality of I and the convexity of C_i imply that $I \cap \bar{C}_i$ is a point or closed line segment or empty. Remove the points of local non-convexity of S from I to get a set I' . Note that we remove at most n sets, each of which must be a closed line segment or a point. Thus I' has at most $n+1$ components, and the closure of each of these components is an arc satisfying the hypothesis of Lemma 1, and hence is a line segment. Thus I consists of at most $2n+1$ line segments and the theorem follows.

BIBLIOGRAPHY

1. F. A. Valentine, *Local convexity and L_n sets*, Proc. Amer. Math. Soc. **16** (1965), 1305–1310. MR 32 #2976.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHARLOTTE, NORTH CAROLINA 28205 (Current address of Nick M. Stavrakas)

Current address (R. E. Jamison): Department of Mathematics, University of Washington, Seattle, Washington 98195