PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, No. 2, December 1972

A NEW PROOF OF A THEOREM ON QUASITRIANGULAR OPERATORS

GLENN R. LUECKE

ABSTRACT. P. R. Halmos has given a proof of the equivalence of two definitions for quasitriangular operators. A short, elementary proof of this fact is given here.

In his paper *Quasitriangular operators* [**1**], Halmos proved the equivalence of the conditions (Δ_0) and (Δ_2) for operators A on Hilbert space H ($\dim H=\infty$). An operator satisfying (Δ_0) or (Δ_2) is called quasitriangular. The proof that (Δ_0) implies (Δ_2) is trivial. However, Halmos uses a three page proof to show that (Δ_2) implies (Δ_0). The following is a short and completely elementary proof of this fact.

Operator A satisfies condition (Δ_2) if there exists a sequence $\{E_n\}$ of (orthogonal) projections of finite rank such that $E_n\rightarrow I$ (strong topology) and $\|AE_n-E_nAE_n\|\rightarrow 0$. Operator A satisfies condition (Δ_0) if for every projection P of finite rank and for every $\varepsilon>0$ there exists a finite rank projection $E\geqq P$ such that $\|AE-EAE\|<\varepsilon$.

THEOREM (HALMOS). *If A satisfies condition (Δ_2), then A satisfies condition (Δ_0).*

PROOF. Use the notation above and let Q_n be the projection on $E_n(N)$, where $N=P(H)$. Let $\frac{1}{2}>\delta>0$ be given. Since $\dim N<\infty$ and since $E_ng\rightarrow g$ for each $g\in H$, there exists n_0 such that for all $n\geqq n_0$, $\|E_ng-g\|<\delta\|g\|$ for all $g\in N$. Let $n\geqq n_0$ and let $f\in E_n(N)$, $\|f\|=1$, $f=E_ng$, $g\in N$. Then $\|g\|\leqq\|g-E_ng\|+\|E_ng\|\leqq\delta\|g\|+1$ so that $\|g\|\leqq(1-\delta)^{-1}$. Then

$$\begin{aligned}\|Q_nf-Pf\| &= \|f-Pf\| = \|E_ng-PE_ng\|\\ &\leqq \|E_ng-Pg\| + \|Pg-PE_ng\|\\ &\leqq \|E_ng-g\| + \|P\|\,\|g-E_ng\|\\ &\leqq (1+\|P\|)\cdot\delta\,\|g\| \leqq 2(1-\delta)^{-1}\delta < 4\delta.\end{aligned}$$

Furthermore if $f\in N$, $\|f\|=1$, then $Pf=f$ and $Q_nf=E_nf$. Hence for $n\geqq n_0$, $\|Q_nf-Pf\|=\|E_nf-f\|<\delta$. Combining this with the previous

Received by the editors April 26, 1972 and, in revised form, June 9, 1972.
AMS (MOS) subject classifications (1970). Primary 47B99.
Key words and phrases. Operators, Hilbert space, quasitriangular operators.

statement we have that, for each $n \geqq n_0$, $\|Q_n f - Pf\| < 4\delta$ for all $f \in N \cup E_n(N)$, $\|f\| = 1$. If $f \in (N \cup E_n(N))^{\perp} \subseteq N^{\perp} \cap (E_n(N))^{\perp}$, then $Q_n f = Pf = 0$. Taking the supremum (for each fixed $n \geqq n_0$) of $\|Q_n f - Pf\|$ over all $\|f\| = 1$, we obtain $\|Q_n - P\| \leqq 4\delta$. Thus $\|Q_n - P\| \to 0$.

Define O_n so that $E_n(H) = E_n(N) \oplus O_n$ and let P_n be the projection on $N \oplus O_n$. Then P_n has finite rank, $P_n \geqq P$ and, since Q_n is the projection on $E_n(N)$, $\|E_n - P_n\| = \|Q_n - P\| \to 0$. Thus since $\|AE_n - E_n A E_n\| \to 0$ and $\|E_n - P_n\| \to 0$, we obtain $\|AP_n - P_n A P_n\| \to 0$. Therefore condition (Δ_0) holds.

Reference

1. P. R. Halmos, *Quasitriangular operators*, Acta Sci. Math. (Szeged) **29** (1968), 283–293. MR **38** #2627.

Department of Mathematics, Iowa State University, Ames, Iowa 50010