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A THEORY OF GRADE FOR COMMUTATIVE RINGS[1]

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ABSTRACT. A theory of grade is developed using R-sequences, the Koszul complex and standard homological algebra. Most results of interest are for finitely generated ideals.

The theory of grade as developed in [5] or [7] is restricted to Noetherian rings. In all that follows rings are commutative with identity and modules are unitary; however no chain conditions are assumed. This theory has been applied in [1] to generalize [6].

Three notions of the grade of an ideal appear in the non-Noetherian case. These are denoted c gr, k gr, and r gr as defined later. A slightly stronger version of the following will be established.

THEOREM. *If I is a finitely generated ideal and E is an R-module with $IE\neq E$, then* $\mathrm{c\,gr}(I, E)\leqq \mathrm{k\,gr}(I, E)\leqq \mathrm{r\,gr}(I, E)$.

1. **Koszul complex and grade.** Let I be an ideal in R and E an R-module with $IE\neq E$. Let $\mathrm{c\,gr}(I, E)$ be the length of the longest maximal R-sequence on E in I. For $\{x_1, \cdots, x_s\}$ a finite subset of I, let $g(x_1, \cdots, x_s|E)=s-t$ where t is the largest integer so that the tth homology module of the Koszul complex over E determined by $x_1, \cdots, x_s$ is not zero. Let $\mathrm{k\,gr}(I, E)$ be the supremum over all finite subsets $\{x_1, \cdots, x_n\}$ of the integers $g(x_1, \cdots, x_n|E)$. For R Noetherian, $\mathrm{c\,gr}(I, E)=\mathrm{k\,gr}(I, E)$. The common value is the grade of I on E. The first proposition follows from the properties of the Koszul complex [7].

PROPOSITION 1. *Let I and J be ideals, E an R-module with $JE\neq E$. Then*
(1) *If $I\subseteq J$*, $\mathrm{k\,gr}(I, E)\leqq \mathrm{k\,gr}(J, E)$.
(2) *If $(x_1, \cdots, x_n)=J$ then* $\mathrm{k\,gr}(J, E)\leqq n$.
(3) $\mathrm{c\,gr}(J, E)\leqq \mathrm{k\,gr}(J, E)$.

An immediate consequence of Proposition 1 is that the length of an R-sequence in an ideal I is bounded by the minimum number of generators

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of I. It is easy to see that for I finitely generated $\mathrm{k\,gr}(I, E)=0$ is equivalent to the statement: there exists $l_0 \neq 0$, l_0 in E, so that $Il_0=0$.

For u in I, u not a zero divisor on E, one can easily verify that $\mathrm{c\,gr}(I, E/uE) \leqq \mathrm{c\,gr}(I, E)-1$. If the quantities involved are finite, one has equality if and only if u is the first term of a longest maximal R-sequence on E in I. By applying Proposition 2 and Theorem 4 of Chapter 8 of [7] one gets $\mathrm{k\,gr}(I, E)=\mathrm{k\,gr}(I, E/uE)+1$. The behaviors under polynomial extensions are covered in the next two propositions.

PROPOSITION 2. *Let S be a faithfully flat extension of R, I an ideal on R, E an R-module with $IE \neq E$, then*
(1) $(I \otimes_R S)(E \otimes_R S) \neq (E \otimes_R S)$, *and*
(2) $\mathrm{k\,gr}(I, E)=\mathrm{k\,gr}(I \otimes_R S, E \otimes_R S)$.

PROOF. The key observation is that for $v_1, \cdots, v_n$ in $I \otimes_R S$, there exist $t_1, \cdots, t_m$ in I so that $(v_1, \cdots, v_n) \subseteq (t_1 \otimes 1, \cdots, t_m \otimes 1)$. The remainder of the proof follows from properties of the Koszul complex and faithfully flat modules [2].

If $\mathrm{c\,gr}(I, E)=0$ and $\mathrm{k\,gr}(I, E) \neq 0$ then for $S=R[x]$ and $a_0, \cdots, a_n$ in I so that $g(a_0, \cdots, a_n|E) \neq 0$, one has $a_0+a_1x+\cdots+a_nx^n$ is not a zero divisor on $E \otimes_R S$. Thus $\mathrm{c\,gr}(I \otimes_R S, E \otimes_R S)>0$. As an extension of this one has:

PROPOSITION 3. *If* $\mathrm{c\,gr}(I, E)=V<n=\mathrm{k\,gr}(I, E)$, *then for some positive integer t and for $S=R[x_1, \cdots, x_t]$ one has*

$$\mathrm{c\,gr}[I \otimes_R S, E \otimes_R S] = \mathrm{k\,gr}[I \otimes_R S, E \otimes_R S] = n.$$

For $I \subseteq J \subseteq \mathrm{Rad}\, I$, it is easy to see that $\mathrm{k\,gr}(I, E)=\mathrm{k\,gr}(J, E)$.

Let k be a field, let u_f be an indeterminate for $f \neq 0$, f in the ideal generated by $x_0, \cdots, x_n$ of $k[x_0, \cdots, x_n]$. Let $T=k[x_0, \cdots, x_n][\{u_f\}]$. Let I_1 and I_2 be ideals generated by objects of the form fu_f and $u_f f_g$ respectively. Let $R=T/(I_1+I_2)$ and $I=(x_0, x_1)$, where x_0 represents $x_0+(I_1+I_2)$. $\mathrm{c\,gr}(I, R)=0$ and $\mathrm{k\,gr}(I, R) \neq 0$. Let $K(a_1, \cdots, a_n)$ denote the Koszul complex determined by $a_1, \cdots, a_n$, and let $H_i(K(a_1, \cdots, a_n))$ denote the ith homology module of $K(a_1, \cdots, a_n)$. One has

$$H_0(K(a_1, \cdots, a_n)) = R/(a_1, \cdots, a_n),$$

and the exact sequence:

$$0 \longrightarrow H_1(K(x_0, x_1)) \longrightarrow R/(x_0) \xrightarrow{x_1} R/(x_0) \longrightarrow R/(x_0, x_1) \longrightarrow 0.$$

Since x_1 is a zero divisor on $R/(x_0)$, $H_1K(x_0, x_1) \neq 0$. So $\mathrm{k\,gr}(I, R)=1$. In like manner one can show $\mathrm{k\,gr}((x_0, x_1, \cdots, x_n), R)=1$.

For another example, let k be a field and $T=k[y, x_1, \cdots, x_n]$. For $f\neq 0$, f a polynomial in $x_1, x_2, \cdots, x_n$ without constant term, adjoin an indeterminate u_f to T forming $T_1=T[\{u_f\}]$. Let I_1 and I_2 be ideals generated by elements of the forms $u_f u_g$ and $u_f(y-f)$ respectively. Let $R=T_1/(I_1+I_2)$ and let $I=(y, x_1, \cdots, x_n)$. $n\leqq \mathrm{c\,gr}(I, R)\leqq \mathrm{k\,gr}(I, R)\leqq n+1$ so $n-1\leqq \mathrm{k\,gr}(I, R/(y))\leqq n$ but $\mathrm{c\,gr}(I, R/(y))=0$.

2. **Homological representation.** Rees [8] and [9] proved that in a Noetherian ring R, $\mathrm{c\,gr}(I, E)=n$ provided n is the least integer so that $\mathrm{Ext}^n(R/I, E)\neq 0$. Denote by $\mathrm{r\,gr}(I, E)$ the least integer n such that $\mathrm{Ext}^n_R(R/I, E)\neq 0$. Rees [9] proves $\mathrm{c\,gr}(I, E)\leqq \mathrm{r\,gr}(I, E)$ and for u in I, u not a zero divisor on E, $\mathrm{r\,gr}(I, E)=1+\mathrm{r\,gr}(I, E/uE)$.

It is easy to see that $\mathrm{r\,gr}(I, E)=0$ is equivalent to the statement: There exist $l_0\neq 0$, l_0 in E, so that $Il_0=0$. For I finitely generated both are equivalent to $\mathrm{k\,gr}(I, E)=0$. For I finitely generated and $\mathrm{c\,gr}(I, E)=\mathrm{k\,gr}(I, E)$ one can easily see that $\mathrm{r\,gr}(I, E)=\mathrm{k\,gr}(I, E)$. This is not true for infinitely generated ideals. For k a field and

$$R = k[x_1, x_2, \cdots, x_n, \cdots]/(x_1^1, x_2^2, x_3^3, \cdots, x_n^n, \cdots)$$

and $I=(x_2, x_3, \cdots, x_n, \cdots)$, $\mathrm{c\,gr}(I, R)=\mathrm{k\,gr}(I, R)=0$ but $\mathrm{r\,gr}(I, R)>0$.

PROPOSITION 4. *For I finitely generated, one has* $\mathrm{k\,gr}(I, E)=n$ *where n is the least integer such that* $\mathrm{Ext}^n_R(R/I, E\infty)\neq 0$. *Here $E\infty$ is the direct sum of countably many copies of E. Further* $\mathrm{k\,gr}(I, E)\leqq \mathrm{r\,gr}(I, E)$.

PROOF. The last remark follows from the first and $E\infty=E\oplus E\infty$. By Proposition 3, there is an integer s so that for $S=R[x_1, \cdots, x_s]$ one has

$$\begin{aligned}\mathrm{k\,gr}(I, E) &= \mathrm{k\,gr}[(I\otimes_R S), (E\otimes_R S)]\\ &= \mathrm{c\,gr}(I\otimes_R S, E\otimes_R S) = \mathrm{r\,gr}(I\otimes_R S, E\otimes_R S).\end{aligned}$$

By the homological identity

$$\mathrm{Ext}^n_R(R/I, E\otimes_R S) \cong \mathrm{Ext}^n_S[S/I\otimes_R S, E\otimes_R S],$$

the result follows. For a proof of the identity see Theorem 3.1 of [3].

A ring R is called *coherent* if every finitely generated ideal is finitely presented. Properties of coherent rings are discussed in [4]. The next proposition was suggested to me by M. Hockster who gave a proof of the first part. As a preliminary, note that for R a ring and k an integer $0\to K_1\to R^k\to K\to 0$ exact, the map $\mathrm{Hom}(R^k, E\infty)\to \mathrm{Hom}(K, E\infty)$ assigns to α in $\mathrm{Hom}(R^k, E\infty)$ a certain matrix map in $\mathrm{Hom}(K_1, E)$.

PROPOSITION 5. *For I finitely generated*:

(1) *If* $\mathrm{k\,gr}(I, E)=1$, *then* $\mathrm{r\,gr}(I, E)=1$.

(2) *If R is coherent and* $\mathrm{k\,gr}(I, E)=n$ *then* $\mathrm{r\,gr}(I, E)=n$.

PROOF. For I finitely generated and R coherent one has a long exact sequence

$$0 \to K \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to I \to 0$$

with K finitely generated and F_i free and finitely generated. This gives

$$0 \to K \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to R \to R/I \to 0.$$

One has $\mathrm{Ext}^n(R/I, E\infty) \neq 0$ where $n = \mathrm{k}\ \mathrm{gr}(I, E)$. If $\mathrm{r}\ \mathrm{gr}(I, E) \neq n$ then one has $\mathrm{Ext}^n(R/I, E) = 0$. Thus $\mathrm{Ext}^1(K, E) = 0$ and $\mathrm{Ext}^1(K, E\infty) \neq 0$. For $F_{n-1} = R^k$, and $0 \to K' \to F^k \to K \to 0$ one gets, for K' finitely generated,

$$\mathrm{Hom}(R^k, E) \xrightarrow{t} \mathrm{Hom}(K', E) \longrightarrow \mathrm{Ext}(K, E) = 0.$$

Thus t is onto. Now, this says every homomorphism $K' \to E$ is given by the map $(v_1, \cdots, v_k) \to \sum v_i l_i$ for $(l_1, \cdots, l_k)$ is a fixed element of E^k. Likewise

$$\mathrm{Hom}(F^k, E\infty) \xrightarrow{\gamma} \mathrm{Hom}(k', E\infty) \longrightarrow \mathrm{Ext}(K, E\infty) \longrightarrow 0$$

so γ is not onto, thus there is a homomorphism $g: K' \to E\infty$ which is not a matrix map. Since K' is finitely generated, $\mathrm{Im}\, g \subseteq E^m$ some m, so g is determined by its projections g_j, $g_j: K' \to E$ by $g_j = g \circ \pi_j$, $\pi_j: E\infty \to E$ (jth projection) $j = 1, \cdots, m$. Thus g_j is determined by $(l_1^{(j)}, l_2^{(j)}, \cdots, l_k^{(j)})$. Thus g is given by a matrix. It is noted that for part (1) one does not need that R is coherent.

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