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PRIMITIVE GROUP RINGS

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ABSTRACT. Two theorems showing the existence of primitive group rings are proved.

THEOREM 1. *Let G be a countable locally finite group and F a field of characteristic* 0, *or characteristic p if G has no elements of order p. Then the group ring $F[G]$ is primitive if and only if G has no finite normal subgroups.*

THEOREM 2. *Let G be any group, and F a field. Then there is a group H containing G such that $F[H]$ is a primitive ring.*

All rings will be associative and have a unit. R is a prime ring if $xRy\neq 0$ whenever x and y are nonzero elements of R. R is a (left) primitive ring if there is a faithful irreducible (left) R-module. Every primitive ring is prime, but not conversely. A group is locally finite if every finitely generated subgroup is finite.

The prime group rings have been completely characterized by the following result:

THEOREM 1 (CONNELL [**1**, p. 675]). *The group ring $R[G]$ is prime if and only if R is a prime ring and G has no finite normal subgroups.*

Very little is known about primitivity in group rings. Almost no progress has been made toward answering the general question, "When is $R[G]$ primitive?" posed by Kaplansky [**2**] and Passman [**3**, p. 136]. Some negative statements are easy to make; for example, if R is a field and $G\neq 1$ is abelian or finite then $R[G]$ is not primitive. Other negative results have been obtained by Alan Rosenberg [**4**]. But there were no examples of primitive $R[G]$ with $G\neq 1$. This paper proves two theorems in a positive direction.

THEOREM 2. *Suppose G is a countable locally finite group and F is a field of characteristic* 0, *or characteristic p if G has no elements of order p. Then $F[G]$ is primitive if and only if it is prime.*

THEOREM 3. *Suppose G is a group and F is a field. Then there is a group H containing G such that $F[H]$ is primitive.*

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Together with Connell's theorem, Theorem 2 provides a class of primitive group rings, while Theorem 3 says there are a lot of primitive group rings. The motivation for studying $F[G]$ for G locally finite (and F of acceptable characteristic) was provided by the following observation of Lance Small: $F[G]$ is von Neumann regular, so if G has no finite normal subgroups, $F[G]$ is either

(1) a group ring which is primitive, or

(2) a prime von Neumann regular ring which is not primitive.

That is to say, $F[G]$ is something new! We are indebted to Susan Montgomery for communicating Small's observation.

THEOREM 4. *The following are equivalent for any ring R.*

(a) *R is left primitive.*

(*b*) *R has a maximal left ideal which contains no nonzero two-sided ideals.*

(c) *R has a proper left ideal A such that $A+B=R$ for every nonzero two-sided ideal B of R.*

The simple proof of Theorem 4 is left to the reader. Criterion (c) will be used in the proof of the next theorem.

A ring R is Artinian semisimple if it is a finite direct product of complete matrix rings over division rings. Such an R has a finite set $e_1, \cdots, e_k$ of irreducible central idempotents relative to which R is an (internal) direct product of simple factors $R=Re_1\times\cdots\times Re_k$.

Theorem 2 is an immediate corollary of the following purely ring theoretic result, taking $R=F[G]$, $R_i=F[G_i]$, where G is expressed as the union of an ascending sequence of finite groups $G_1\subset G_2\subset G_3\cdots$.

THEOREM 5. *Suppose R is the union of an ascending sequence $R_1\subset R_2\subset R_3\cdots$ of Artinian semisimple rings. Then R is primitive if and only if it is prime.*

PROOF. Since all primitive rings are prime, it is enough to show that if R is prime, it is primitive.

Let $\{e_1, e_2, \cdots\}$ be an enumeration of all the irreducible central idempotents of all the R_i. It is worth noting that the e_i commute with each other. Define a sequence of pairs (R_{n_1}, f_1), $(R_{n_2}, f_2), \cdots$, where f_i is an irreducible central idempotent of R_{n_i}, inductively as follows.

Initial step. Let $f_1=e_1$ and R_{n_1} be an R_i in which e_1 is an irreducible central idempotent.

Inductive step. Assume $(R_{n_1}, f_1), \cdots, (R_{n_k}, f_k)$ have been chosen so that the following three conditions are satisfied:

(1) $e_1\in R_{n_1}, e_2\in R_{n_2}, \cdots, e_k\in R_{n_k}$.

(2) $e_1f_1\neq 0, e_2f_2\neq 0, \cdots, e_kf_k\neq 0$.

(3) $f_1\cdots f_k\neq 0$.

Then choose $(R_{n_{k+1}}, f_{k+1})$ as follows: Using the fact that R is a prime ring choose $r \in R$ such that $e_{k+1} r f_1 \cdots f_k \neq 0$, next, choose n_{k+1} so large that e_{k+1}, $r \in R_{n_{k+1}}$; finally, let f_{k+1} be an irreducible central idempotent of $R_{n_{k+1}}$ such that $(e_{k+1} r f_1 \cdots f_k) f_{k+1} \neq 0$. Since f_{k+1} commutes with $r f_1 \cdots f_k$ it follows that $e_{k+1} f_{k+1} \neq 0$, $f_1 \cdots f_{k+1} \neq 0$ and the three conditions are again satisfied for $k+1$.

Now let A be the left ideal of R generated by $\{1-f_1, 1-f_2, \cdots\}$.

(1) A is a proper ideal of R.

If not, there is an integer k and $r_1, \cdots, r_k \in R$ such that

$$1 = r_1(1-f_1) + \cdots + r_k(1-f_k). \tag{$*$}$$

$f = f_1 \cdots f_k \neq 0$ and $(1-f_i)f = 0$ for $i = 1, \cdots, k$ since the f_i are commuting idempotents. Multiplying $(*)$ on the right by f yields the contradiction $f = 1 \cdot f = 0$.

(2) If B is a nonzero two-sided ideal of R, $A+B=R$.

For B must contain some e_i and hence some f_i since $e_i f_i \neq 0$ lies in the simple factor $R_{n_i} f_i$ of R_{n_i}.

$$\therefore 1 = (1-f_i) + f_i \in A + B.$$

By Theorem 4, R is primitive.

We conclude with a proof of Theorem 3. The construction we use yields an extravagantly large group H containing the original group G. Passman has pointed out how to modify the construction so that H and G have the same cardinality in case G is infinite. However, the given proof illustrates the idea in the most straightforward fashion.

Proof of Theorem 3. Define a sequence $\{G_i\}$ of groups and a sequence $\{M_i\}$ of modules inductively by

$$\begin{aligned} G_1 &= G, & M_1 &= F[G_1], \\ G_2 &= \mathrm{Aut}_F(M_1), & M_2 &= F[G_2] \oplus M_1, \\ G_3 &= \mathrm{Aut}_F(M_2), & M_3 &= F[G_3] \oplus M_2, \\ &\ \ \vdots & &\ \ \vdots \end{aligned}$$

$G_1 \subset G_2 \subset G_3 \cdots$, $M_1 \subset M_2 \subset M_3 \cdots$, and we let $H = \bigcup G_i$, $M = \bigcup M_i$. Each M_i is an $F[G_i]$-module via the obvious action and this action makes M an $F[H]$-module. Moreover, M is a faithful, irreducible $F[H]$-module, which shows that $F[H]$ is primitive. M is faithful because each M_i is a faithful $F[G_i]$-module. M is irreducible since each M_i is an irreducible $F[G_{i+1}]$-module. (In fact, G_{i+1} acts transitively on the nonzero elements of M_i, so H acts transitively on the nonzero elements of M.)

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