PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, No. 2, December 1972

TIME DEPENDENT NONLINEAR CAUCHY PROBLEMS IN BANACH SPACES

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ABSTRACT. The method of product integration is used to obtain solutions to the nonlinear evolution equation $u'(t)+A(t)u(t)=0$ where $\{A(t):t\in[0, T]\}$ is a family of nonlinear accretive operators mapping a Banach space X to itself. The main requirements are that $R(I+\lambda A(t))\supseteq \mathrm{cl}(D(A(t)))$, $D(A(t))$ is time independent, the resolvent $(I+\lambda A(t))^{-1}x$ satisfies a local Lipschitz condition, and that each $A(t)$ satisfies Condition M.

This paper is concerned with the existence of solutions to the evolution equation

$$u'(t) + A(t)u(t) = 0; \qquad u(0) = x,$$

where $A(t)$ is a nonlinear accretive operator on a Banach space. This equation has been the subject of much activity within the last five years. Several authors have studied existence theory (e.g. T. Kato ([**6**], [**7**]), J. Mermin ([**10**], [**11**]), R. Martin ([**8**], [**9**]), V. Barbu [**1**], H. Brezis and A. Pazy [**2**], M. Crandall and T. Liggett [**3**], and G. Webb ([**13**], [**15**])).

The author is grateful for the opportunity to see preprints of the aforementioned manuscripts of Crandall and Liggett and Brezis and Pazy. Special appreciation is due G. F. Webb for suggesting the problems contained herein. The author wishes to thank the referee for his helpful suggestions and criticisms.

DEFINITION 1.1. An operator A is said to be accretive provided that $\|(x+\lambda Ax)-(y+\lambda Ay)\|\geqq\|x-y\|$ for $x, y\in D(A)$ and $\lambda\geqq 0$. Kato [**7**] has shown that this definition is equivalent to the statement that

$$\mathrm{Re}(Ax - Ay, f) \geqq 0$$

for some $f\in F(x-y)$ where F is the duality map from X to X^*.

It is clear that $(I+\lambda A)^{-1}$ is a function having domain $D_\lambda=R(I+\lambda A)$. We denote $J_\lambda x=(I+\lambda A)^{-1}x$ for $\lambda>0$ and let $A_\lambda x=\lambda^{-1}(I-J_\lambda)x$. The

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Presented to the Society, November 20, 1971; received by the editors November 18, 1971 and, in revised form, March 6, 1972.

AMS (MOS) subject classifications (1969). Primary 3495, 3436; Secondary 3535, 3537.

Key words and phrases. Nonlinear evolution equation, product integration, accretive operator.

following facts are well known:

$$
(1.2)\qquad
\begin{aligned}
\|J_\lambda x - J_\lambda y\| &\leqq \|x - y\|. \\
\|J_\lambda x - x\| &\leqq \lambda\, \|Ax\| && \text{for } x \in D_\lambda \cap D(A). \\
AJ_\lambda x &= A_\lambda x && \text{for } x \in D_\lambda. \\
\|A_\lambda x\| &\leqq \|Ax\| && \text{for } x \in D_\lambda \cap D(A).
\end{aligned}
$$

The following result is due to Crandall and Liggett [3]:

THEOREM A. *Let* $\{A(t): t \in [0, T]\}$ *be a family of accretive operators. Assume that the following conditions hold*:

(1) $D(A(t))$ *is independent of* t.

(2) $R(I+\lambda A(t)) \supseteq \mathrm{cl}(D(A(t)))$ *for* $0<\lambda\leqq\lambda_0$ *and* $t \in [0, T]$.

(3) $\|A(t)x\| \leqq \|A(\tau)x\| + |t+\tau| L(\|x\|)(1+\|A(\tau)x\|)$ *for* $t, \tau \in [0, T]$ *and* $x \in D(A(0))$.

(4) $\|(I+\lambda A(t))^{-1}x-(I+\lambda A(\tau))^{-1}x\| \leqq \lambda |t-\tau| L(\|x\|+\|A(\tau)x\|)$ *for* $t, \tau \in [0, T]$ *and* $x \in D(A(0))$.

Here $L: [0, \infty) \to [0, \infty)$ *is an increasing function. Then*

$$u(t) = \lim_{n\to\infty} \prod_{i=1}^{[t/\varepsilon n]} (I + \varepsilon_n A(i\varepsilon_n))^{-1} x$$

exists for $x \in D(A(0))$, $\varepsilon_n \downarrow 0$, *such that* $\varepsilon_n \leqq T/n$, *and* $t \in [0, T]$.

We shall refer to the above limit as a product integral.

In the course of the proof of Theorem A one establishes the existence of a constant $M(x)$ such that

$$\left\| \prod_{i=1}^{l} (I + \varepsilon_n A(i\varepsilon_n))^{-1} x \right\| \leqq M(x), \tag{1.3}$$

$$\left\| A(u) \prod_{i=1}^{l} (I + \varepsilon_n A(i\varepsilon_n))^{-1} x \right\| \leqq M(x), \tag{1.4}$$

whenever $0 \leqq l \leqq [t/\varepsilon_n]$, ε_n is sufficiently small, and $u, t \in [0, T]$.

DEFINITION 1.5. An operator A is said to satisfy Condition M provided that whenever $\{x_n\} \subseteq D(A)$, $x_n \to x_0$, and $\sup \|Ax_n\| < \infty$, then $x_0 \in D(A)$ and $Ax_0 = w - \lim Ax_n$. This condition was introduced by R. Martin in [9].

DEFINITION 1.6. A function $u: [0, T) \to X$ is a strong solution to the Cauchy problem

$$du(t)/dt + A(t)u(t) = 0, \qquad u(0) = x, \tag{1.7}$$

provided that u is Lipschitz continuous on each compact subset of $[0, T)$, $u(0)=x$, u is strongly differentiable almost everywhere, and $du(t)/dt+A(t)u(t)=0$ for a.e. $t \in [0, T)$.

THEOREM 1.8. *Let $\{A(t): t \in [0, T]\}$ be a family of accretive operators satisfying Condition* M *such that the following are true:*

(1) *$D(A(t))$ is independent of t.*

(2) *$R(I+\lambda A(t)) \supseteq \mathrm{cl}(D(A(t)))$.*

(3) *$\|(I+\lambda A(t))^{-1}x-(I+\lambda A(\tau))^{-1}x\| \leqq \lambda|t-\tau| L(\|x\|)(1+\|A(\tau)x\|)$, where $t, \tau \in [0, T]$, $\lambda>0$ and $L: [0, \infty) \to [0, \infty)$ is an increasing function.*

Then the Cauchy problem (1.7) *has an unique strong solution for $x \in D(A(0))$ on $[0, T)$.*

PROOF. To ascertain that the conditions of Theorem A are met we modify L by defining $L^1(u)=L(1+u)(1+u)$. Thus,

$$\begin{aligned} &\|(I + \lambda A(t))^{-1}x - (I + \lambda A(\tau))^{-1}x\| \\ &\qquad \leqq \lambda\,|t - \tau|\, L(\|x\|)(1 + \|A(\tau)x\|) \\ &\qquad \leqq \lambda\,|t - \tau|\, L(1 + \|x\| + \|A(\tau)x\|)(1 + \|x\| + \|A(\tau)x\|) \\ &\qquad \leqq \lambda\,|t - \tau|\, L^1(\|x\| + \|A(\tau)x\|). \end{aligned}$$

To see that the third condition of Theorem A is satisfied, we observe that Condition M together with (1.2) yield $A_\lambda(t)x \rightharpoonup A(t)x$ as $\lambda \downarrow 0$. Thus

$$\begin{aligned} \big|\, \|A(t)x\| - \|A(\tau)x\| \,\big| &\leqq \|A(t)x - A(\tau)x\| \\ &\leqq \liminf_{\lambda \downarrow 0} \|A_\lambda(t)x - A_\lambda(\tau)x\| \\ &\leqq |t - \tau|\, L(\|x\|)(1 + \|A(\tau)x\|) \\ &\leqq |t - \tau|\, L^1(\|x\|)(1 + \|A(\tau)x\|). \end{aligned} \tag{1.9}$$

Hence the conditions of Theorem A are met and the product integral

$$u(t) = \lim_{n\to\infty} \prod_{1}^{[t/\varepsilon_n]} (I + \varepsilon_n A(i\varepsilon_n))^{-1}x$$

exists for $t \in [0, T]$ and for $\varepsilon_n \downarrow 0$ such that $\varepsilon_n \leqq T/n$.

We now define a sequence of step functions,

$$\begin{aligned} u_{\varepsilon_n}(t) &= \prod_{i=1}^{[t/\varepsilon_n]} (I + \varepsilon_n A(i\varepsilon_n))^{-1}x \quad \text{if } t \geqq 0, \\ &= x \quad \text{if } t < 0. \end{aligned} \tag{1.10}$$

By observing that

$$\begin{aligned} (u_{\varepsilon_n}(t) - u_{\varepsilon_n}(t - \varepsilon_n))/\varepsilon_n &= -A_{\varepsilon_n}([t/\varepsilon_n]\varepsilon_n)u_{\varepsilon_n}(t - \varepsilon_n) \\ &= -A([t/\varepsilon_n]\varepsilon_n)u_{\varepsilon_n}(t), \end{aligned}$$

we obtain a solution to the approximate equation

$$(1.11)\quad (u_{\varepsilon_n}(t) - u_{\varepsilon_n}(t-\varepsilon_n))/\varepsilon_n + A([t/\varepsilon_n]\varepsilon_n)u_{\varepsilon_n}(t) = 0 \quad \text{and} \quad u_{\varepsilon_n}(0) = x.$$

If we apply any linear functional $f \in X^*$ to (1.11) and integrate over $(0, t)$ we obtain the equation,

$$\int_{t-\varepsilon_n}^{t} (u_{\varepsilon_n}(s), f)\, ds - (x, f) + \int_0^t (A([s/\varepsilon_n]\varepsilon_n)u_{\varepsilon_n}(s), f)\, ds = 0; \qquad u_{\varepsilon_n}(0) = x.$$

From the above equation we obtain

$$(1.12)\quad \begin{aligned} \int_{t-\varepsilon_n}^{t} (u_{\varepsilon_n}(s), f)\, ds &= (x, f) - \int_0^t (A(s)u_{\varepsilon_n}(s), f)\, ds \\ &\quad + \int_0^t (A(s)u_{\varepsilon_n}(s) - A([s/\varepsilon_n]\varepsilon_n)u_{\varepsilon_n}(s), f)\, ds; \\ &\qquad u(0) = x. \end{aligned}$$

Now we apply (1.9) to see that

$$\begin{aligned} &\left| \int_0^t (A(s)u_{\varepsilon_n}(s) - A([s/\varepsilon_n]\varepsilon_n)u_{\varepsilon_n}(s), f)\, ds \right| \\ &\quad \leqq \varepsilon_n t \max_{S\in[0,t]} \{ |s/\varepsilon_n - [s/\varepsilon_n]|\, L^1(\|u_{\varepsilon_n}(s)\|)(1 + \|A(s)u_{\varepsilon_n}(s)\|)\}\, \|f\| \leqq \varepsilon_n B(x) \end{aligned}$$

for some constant $B(x) > 0$. Since $u_{\varepsilon_n}(t) \to u(t)$ and $\|A(t)u_{\varepsilon_n}(t)\|$ is bounded, Condition M insures that $A(t)u_{\varepsilon_n}(t) \rightharpoonup A(t)u(t)$. Thus we can take limits of (1.12) as $\varepsilon_n \downarrow 0$ to derive the equation

$$(1.13)\qquad (u(t), f) = (x, f) - \int_0^t (A(s)u(s), f)\, ds; \qquad u(0) = x.$$

We can verify the strong measurability of $A(t)u(t)$ by observing that $A(t)u(t)$ is the weak pointwise limit of a sequence of piecewise continuous functions $A(t)u_{\varepsilon_n}(t)$. Thus $A(t)u(t)$ is strongly measurable and bounded, hence, Bochner integrable.

From (1.13) we obtain the equation

$$u(t) = u(0) - \int_0^t A(s)u(s)\, ds$$

and hence that

$$du(t)/dt + A(t)u(t) = 0 \quad \text{for a.e. } t \in [0, T).$$

To see that accretiveness guarantees uniqueness we apply the methods of Kato [7]. Suppose $u(t)$ and $v(t)$ are solutions satisfying the initial conditions $u(0)=a$ and $v(0)=b$. Set $x(t)=u(t)-v(t)$, then $\frac{1}{2}d\|x(t)\|^2/dt=-\mathrm{Re}(A(t)u(t)-A(t)v(t),f)\leqq 0$, where $f\in F(x(t))$. Since $\|x(t)\|^2$ is absolutely continuous $\|x(t)\|^2\leqq\|x(0)\|^2=\|a-b\|^2$.

Two immediate corollaries follow from Theorem 1.8.

COROLLARY 1.14. *Let X be a reflexive Banach space and let $\{A(t):t\in[0,T)\}$ be a family of demiclosed accretive operators which satisfy* (1)–(3) *of the theorem. Then the Cauchy problem* (1.7) *has a unique strong solution.*

PROOF. We need only observe that a demiclosed accretive operator in a reflexive Banach space satisfies Condition M.

COROLLARY 1.15. *Let $\{A(t):t\in[0,T]\}$ be a family of accretive operators such that $A(t)$ is continuous from the strong to the weak topology of X. If $A(t)$ satisfies conditions* (1)–(3) *with the added condition that $D(A(t))$ is closed, the Cauchy problem* (1.7) *has a strong solution.*

Theorem 1.8 may be applied to time dependent perturbations of linear equations.

THEOREM 1.16. *Let A be a closed, densely defined m-accretive linear operator from a reflexive Banach space to itself. For each $t\in[0,T]$, let $B(t)$ be a continuous, nonlinear, everywhere defined accretive operator from X to itself such that $B(t)0=0$ for all $t\in[0,T]$. We further require that $B(t)$ satisfies the local Lipschitz condition*

$$\|B(t)x-B(\tau)x\|\leqq|t-\tau|\,L(\|x\|)$$

where L is an increasing function $L:[0,\infty)\to[0,\infty)$. Then there exists a unique strong solution to the perturbed equation,

$$du(t)/dt+Au(t)+B(t)u(t)=0.$$

PROOF. In [**16**], Webb shows that for $t\in[0,T]$ the operator $A+B(t)$ is m-accretive, i.e., $R(I+\lambda(A+B(t)))=X$ for $\lambda\geqq 0$. The local Lipschitz condition on $B(t)$ provides the local Lipschitz condition on the resolvent. Thus, by virtue of Corollary 1.14, we need only show that A is demiclosed. Let $\{x_n\}\subset D(A)$ and suppose that $x_n\to x$ and $A(x_n)\rightharpoonup y$. By a theorem of Mazur there exists a finite set of real numbers $\{a_i\}$, $a_i\geqq 0$, $\sum a_i=1$, such that $\sum a_iA(x_{n+i})\to y$ and hence $A(\sum a_ix_{n+i})\to y$. However $\sum a_ix_{n+i}\to x$ and thus the closedness of A implies $Ax=y$.

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