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COMPLEXES WITH THE DISAPPEARING CLOSED SET PROPERTY

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ABSTRACT. A topological space X is said to have the disappearing closed set (DCS) property if and only if for every proper closed subset C there is a sequence of homeomorphisms $\{h_i\}$, $i=1, 2, 3, \cdots$, of X onto X, and a decreasing sequence of open subsets $\{U_i\}$, $i=1, 2, 3, \cdots$, of X such that $\bigcap_{i=1}^{\infty} U_i=\varnothing$ and $h_i(C)\subseteq U_i$. THEOREM. *A finite simplicial n-complex is an n-manifold if and only if it has the DCS property.*

A topological space X is said to have the disappearing closed set (DCS) property if and only if for every proper closed set $C\subset X$ there is a decreasing family of open sets $\{U_i\}$, $i=1, 2, \cdots$, in X such that $\bigcap_{i=1}^{\infty} U_i=\varnothing$, and a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ of X onto X such that $h_i(C)\subseteq U_i$, $i=1, 2, \cdots$.

This definition was motivated by an attempt to study ideas related to invertible spaces ([**1**], [**2**]). Examples of spaces with the DCS property are the n-sphere, open n-cell, torus and open annulus. A disconnected example is $(0, 1)\cap\{\text{rationals}\}$. It can be proved that any disconnected DCS space must have infinitely many components, and that the product of two DCS spaces will also have the DCS property [**3**]. It is proved here that, under certain restrictions, the union of two spaces each of which has the DCS property will have the DCS property. This is then used to prove that a finite simplicial n-complex is an n-manifold if and only if it has the DCS property.

THEOREM 1. *Let X be a space with two intersecting open subspaces C and D each of which has the DCS property, such that for any proper closed subset of C (or D, resp.), the sequences of open sets $\{U_i\}_{i=1}^{\infty}$ ($\{V_i\}_{i=1}^{\infty}$) and homeomorphisms $\{h_i\}_{i=1}^{\infty}$ ($\{k_i\}_{i=1}^{\infty}$) may be taken to have the following properties:*

(a) *h_i (k_i) may be extended to a homeomorphism of X onto X that is the identity on $X-C$ ($X-D$).*

(b) *There is a positive integer M (N) such that $C\cap(D-\bar{U}_M)\neq\varnothing$ ($D\cap(C-\bar{V}_N)\neq\varnothing$). (The closure is with respect to $C\cup D$.)*

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Then $C \cup D$ has the DCS property, with a sequence of homeomorphisms $\{\varphi_i\}_{i=1}^{\infty}$ such that φ_i may be extended to a homeomorphism of X onto X that is the identity on $X-(C\cup D)$.

PROOF. Let A be a proper closed set in $C\cup D$, with respect to the relative topology. If $A\cap C=\varnothing$ or $A\cap D=\varnothing$, there is nothing to prove. Without loss of generality, it may be assumed that $A\cap C$ is a proper closed subset of C. From the DCS property on C and the hypothesis of the theorem, there is a sequence $\{U_i\}_{i=1}^{\infty}$ of open sets in C such that $U_{i+1}\subseteq U_i$ and $\bigcap_{i=1}^{\infty} U_i=\varnothing$, and a sequence of homeomorphisms $\{h_i\}_{i=1}^{\infty}$ of X onto X such that $h_i(A\cap C)\subseteq U_i$ and $h_i|X-C$ is the identity, for all i.

Let M be the positive integer such that $C\cap(D-\bar{U}_M)\neq\varnothing$. $[\bar{U}_M\cap D]\cup(D-C)=B$ is a closed proper subset of D, so there is a sequence $\{k_i\}_{i=1}^{\infty}$ of homeomorphisms of X onto X, the identity on $X-D$ and a decreasing sequence $\{V_i\}_{i=1}^{\infty}$ of open sets in D such that $k_i(B)\subseteq V_i$, for all i, and $\bigcap_{i=1}^{\infty}V_i=\varnothing$.

Let $\varphi_i=k_i\circ h_{M+i}$, and let $W_i=U_{M+i}\cup V_i$. W_i is open for all i. Now $h_{M+i}(A)\subseteq B\cup[U_{M+i}\cap(C-D)]$, so $\varphi_i(A)\subseteq k_i\{B\cup[U_{M+i}\cap(C-D)]\}\subseteq W_i$. (Note that $\varphi_i|C-D=h_{M+i}$, and $\varphi_i|D-C=k_i$.) Also, $W_{i+1}\subseteq W_i$, since $U_{M+i+1}\subseteq U_{M+i}$ and $V_{i+1}\subseteq V_i$, and $\bigcap_{i=1}^{\infty} W_i=\varnothing$. Finally $\varphi_i|X-(C\cup D)$ is the identity, so the theorem is proved.

THEOREM 2. *Let D be a closed n-cell, with interior C. Then C has the DCS property, and the sequences of DCS homeomorphisms may be chosen to extend to the identity on $D-C$.*

PROOF. Let A be a closed proper subset of C in the relative topology for C. There is a point $y\in C-A$ which is contained in an open set in $C-A$. Since D is a closed n-cell, it may be expressed as a closed cone neighborhood of y, say $[0,1)\times S^{n-1}\cup y$. By radial projection on this cone, holding y and $D-C$ pointwise fixed, it is possible to homeomorphically move A into $[0,1/m)\times S^{n-1}$, for all m. If we then take $U_m=(0,1/m)\times S^{n-1}$, the theorem is proved.

COROLLARY (TO THE PROOF OF THEOREM 2). *If B is an open n-cell such that $C\cap B\neq\varnothing$, there is a p such that $C\cap(B-\bar{U}_p)\neq\varnothing$.*

THEOREM 3. *A finite, connected simplicial n-complex is an n-manifold if and only if it has the DCS property.*

PROOF. The set of points where a complex is not locally Euclidean is closed and invariant (and a proper subset). It must be empty by the DCS property. Thus, the complex is an n-manifold.

Now, suppose the n-complex X is an n-manifold. Let $S_1, S_2, \cdots, S_n$ be the (closed) stars of the n-vertices in X. Let U_i be the interior of S_i,

$i=1, \cdots, n$. Note that $X=\bigcup_{i=1}^{n} U_i$. Each U_i is an open n-cell, since X is a manifold.

Let A be a proper closed subset of X. There is a j such that $A \cap U_j$ is a proper relatively closed subset of U_j. Let $U_j = V_1$, and $V_k = U_i$ with i such that $U_i \cap V_{k-1} \neq \varnothing$, $V_k \nsubseteq \bigcup_j^{k-1} V_j$.

Now by Theorem 2, the DCS homeomorphisms for V_k may be taken to be the identity on $\bar{V}_k - V_k = \mathrm{Bd}\ V_k$, so they may be taken to be the identity on $X - V_k$. By Theorems 1 and 2, $V_1 \cup V_2$ has the DCS property.

Similar reasoning shows that $\bigcup_{k=1}^{i} V_k$ has the DCS property, $i=3, \cdots, n$, since $\bigcup_{k=1}^{i-1} V_k$ and V_i satisfy conditions (a) and (b) in Theorem 1. Thus, the theorem is proved.

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