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NONNEGATIVE MATRICES WHOSE INVERSES ARE M-MATRICES

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ABSTRACT. A characterization of a class of totally nonnegative matrices whose inverses are M-matrices is given. It is then shown that if A is nonnegative of order n and A^{-1} is an M-matrix, then the almost principal minors of A of all orders are nonnegative.

I. **Introduction.** Suppose $A=(a_{ij})$ is a matrix of order n. We write $A\geqq 0$ if $a_{ij}\geqq 0$ for each pair (i,j). A is called totally nonnegative (totally positive) if all minors of all orders of A are nonnegative (positive). Finally, if A is totally nonnegative, and a power of A is totally positive, then A is said to be oscillatory (see [**2**], [**3**] for pertinent results).

Fiedler and Pták gave the following characterization of M-matrices in [**1**], which we shall use as a definition.

DEFINITION 1.1. Suppose A is a real $n\times n$ matrix with nonpositive off-diagonal elements. Then A is an M-matrix if and only if A is nonsingular and $A^{-1}\geqq 0$.

In §II, we offer a characterization of a class of totally nonnegative matrices whose inverses are M-matrices. We prove in §III that if $A\geqq 0$ and A^{-1} is an M-matrix, then the almost principal minors of A of all orders are nonnegative.

II. $A\geqq 0$ **with** A **totally nonnegative.** All matrices considered are of order n. Let $A_{i,j}$ be the submatrix of A of order $n-1$ obtained by deleting row i and column j.

THEOREM 2.1. *Suppose A is a nonsingular, totally nonnegative matrix. Then A^{-1} is an M-matrix if and only if* $\det(A_{i,j})=0$ *for* $i+j=2K$, *where K is a positive integer, and* $i\neq j$.

PROOF. Suppose $A^{-1}=(\alpha_{ij})$ is an M-matrix. Then $\alpha_{ij}\leqq 0$ for $i\neq j$. But $\alpha_{ij}=[(-1)^{i+j}\det(A_{j,i})]/\det(A)$. Since A is totally nonnegative, we have $\det(A_{j,i})\geqq 0$ and A nonsingular implies $\det(A)>0$. Thus we have $\det(A_{j,i})=0$ for $i+j=2K$ and $i\neq j$.

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If $\det(A_{j,i})=0$ for $i+j=2K$ and $i\neq j$, then clearly $\alpha_{ij}\leqq 0$ for $i\neq j$. The fact that A^{-1} is an M-matrix now follows from Definition 1.1.

Next, we examine a special class of oscillatory matrices with the property that each element has an M-matrix as its inverse. We call $A=(a_{ij})$ a matrix of type D if

$$a_{ij} = \begin{cases} a_i, & i \leqq j, \\ a_j, & i > j, \end{cases} \quad \text{where } a_n > a_{n-1} > \cdots > a_1.$$

It was shown by the author in **[3]** that $a_1>0$, then a matrix of type D is oscillatory.

THEOREM 2.2. *Suppose A is a matrix of type D with $a_{11}>0$. Then* $\det(A_{i,j})=0$ *for* $|i-j|>1$.

PROOF. Since A is symmetric, we shall assume $j>i+1$. If $i=1$, then the second column of $A_{i,j}$ is a multiple of the first column and $\det(A_{i,j})=0$. If $j=n$, then the last two rows of $A_{i,n}$ are identical and $\det(A_{i,n})=0$. We assume $i\neq 1$ and $j\neq n$. Let

$$\det(A_{i,j}) = \det\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where B_1 is $(i-1)\times(i+1)$ and B_4 is of order $(n-i)\times(n-i-2)$. (Note that $n\geqq 4$ here.) Using the Laplace expansion for $\det(A_{i,j})$ and expanding by the last $n-i$ rows of $A_{i,j}$, we see that 2 columns must always be chosen from B_3 since B_4 contains only $n-i-2$ columns. But in B_3 all columns are multiples of the first column. Thus in the sum of the determinants in the Laplace expansion, each term is zero, and hence $\det(A_{i,j})=0$. The proof is complete.

THEOREM 2.3. *Suppose $A=(a_{ij})$ is a matrix of type D with $a_{11}>0$. Then A^{-1} is a tridiagonal M-matrix.*

PROOF. A^{-1} is tridiagonal, since $\det(A_{i,j})=0$ for $|i-j|>1$, and A^{-1} is an M-matrix by Theorem 2.1 and Definition 1.1.

III. Nonnegativity of almost principal minors of matrices whose inverses are M-matrices. Gantmacher and Kreĭn defined the term *almost principal minor* in their study of totally nonnegative matrices **[2]**. We shall use the following definition: If α and β are strictly increasing sequences on $N=\{1,\cdots,n\}$ of the same length, then $A(\alpha|\beta)$ is the minor of A with rows indexed by α and columns indexed by β. We say that $A(\alpha|\beta)$ is an almost principal minor of A if in the sequence $|\alpha-\beta|=(|\alpha_1-\beta_1|,\cdots,|\alpha_K-\beta_K|)$ exactly one term is nonzero.

Our main result is the

THEOREM 3.1. *If $A \geqq 0$ and A^{-1} is an M-matrix, then the almost principal minors of A are nonnegative.*

First, we prove the

LEMMA. *If*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & a_{nn} \end{pmatrix} \geqq 0$$

where A_{11} is of order $n-1$, and if A^{-1} is an M-matrix, then A_{11}^{-1} exists and is an M-matrix.

PROOF. To demonstrate that A_{11} is nonsingular, we partition A^{-1} conformably with A as

$$A^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & b_{nn} \end{pmatrix} = (b_{ij}).$$

Immediately we obtain the relation

$$A_{11}B_{11} + A_{12}B_{21} = I. \tag{1}$$

Since $A_{12}B_{21}$ is of rank at most one, its characteristic polynomial is

$$\begin{aligned} p(m) &= \det[mI - A_{21}B_{21}] \\ &= m^{n-1} - [\operatorname{trace}(A_{12}B_{21})]m^{n-2}. \end{aligned}$$

Thus $p(1) = \det(I - A_{12}B_{21}) = 1 + \sum_{i=1}^{n-1} a_{in}|b_{ni}| \geqq 1$. This implies that $A_{11}B_{11}$ is nonsingular. Thus both A_{11} and B_{11} are nonsingular, and from (1), we get

$$A_{11}^{-1} = B_{11}(I - A_{12}B_{21})^{-1}. \tag{2}$$

We show next that $C = (I - A_{12}B_{21})^{-1}$ is an M-matrix, and finally that $B_{11}C$ is an M-matrix.

It is easy to verify that $C = (c_{ij})$ where

$$c_{ii} = \left(1 + \sum_{j \neq i} a_{jn}|b_{nj}|\right) \Big/ \det(I - A_{12}B_{21}) \quad \text{for all } i,$$

$$c_{ij} = a_{in}b_{nj}/\det(I - A_{12}B_{21}) \quad \text{for } i \neq j.$$

Hence C has nonpositive off-diagonal elements, and $C^{-1} = (I - A_{12}B_{21}) \geqq 0$. So C is an M-matrix. Also, B_{11} is an M-matrix since A^{-1} is an M-matrix.

Let $d=\det(I-A_{12}B_{21})$. For $i\neq j$, we have

$$\begin{aligned}(A_{11}^{-1})_{i,j} &= \sum_{k=1}^{n-1} b_{ik}c_{kj} \\ &= \frac{1}{d}\left\{\sum_{k\neq j} b_{ik}a_{kn}b_{nj} + b_{ij}\left(1+\sum_{p\neq j} a_{pn}\,|b_{np}|\right)\right\} \\ &= \frac{b_{nj}}{d}\left(\sum_{k\neq j}^{n-1} b_{ik}a_{kn}\right) + \frac{b_{ij}}{d}\left(1+\sum_{p\neq j}^{n-1} a_{pn}\,|b_{np}|\right).\end{aligned}$$

From $BA=I$, we obtain $\sum_{k\neq j}^{n-1} b_{ik}a_{kn}=-b_{ij}a_{jn}-b_{in}a_{nn}\geqq 0$, and so $(A_{11}^{-1})_{i,j}\leqq 0$ for $i\neq j$. A_{11}^{-1} is an M-matrix since $A_{11}\geqq 0$, and the lemma is proved.

There is nothing special about the fact that A_{11} is contained in consecutive rows and columns $1, 2, \cdots, n-1$. For if E is a principal submatrix of $A\geqq 0$ of order $n-1$, we can simultaneously permute rows and columns of A such that

$$PAP^T = \begin{pmatrix} E & E_{12} \\ E_{21} & e_{nn}\end{pmatrix}$$

and $PAP^T\geqq 0$. It is clear that if A^{-1} is an M-matrix, then $PA^{-1}P^T$ is an M-matrix by Definition 1.1. Hence we state the

COROLLARY. *If $A\geqq 0$ and A^{-1} is an M-matrix and if S is a principal submatrix of A of order $n-1$, then S^{-1} exists and is an M-matrix.*

We return to the proof of Theorem 3.1.

The almost principal minors of A of order $n-1$ are nonnegative since $b_{i,i+1}\leqq 0$ and $b_{i+1,i}\leqq 0$ for $i=1,\cdots,n-1$, i.e. $\det(A_{i,i+1})\geqq 0$ and $\det(A_{i+1,i})\geqq 0$ for $i=1,\cdots,n-1$, and these exhaust the almost principal minors of order $n-1$.

Any almost principal minor of order $n-2$ or less is contained in a principal submatrix, S, of A of order $n-1$. The proof is completed by using induction.

The condition of Theorem 3.1 is not sufficient for A^{-1} to be an M-matrix. Suppose

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6\end{pmatrix}.$$

Then $A\geqq 0$ and the almost principal minors of A are nonnegative. In fact, A is oscillatory. A^{-1} is not an M-matrix since $\det(A_{3,1})=1$ and Theorem 2.1 does not hold.

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