PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 36, No. 2, December 1972

# PERIODIC SOLUTIONS OF SMALL PERIOD OF SYSTEMS OF $n$TH ORDER DIFFERENTIAL EQUATIONS[1]

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ABSTRACT. This paper consists of a study of the existence of periodic solutions of systems of $n$th order ordinary differential equations using tools from degree theory.

1. **Introduction.** Let $I$ denote the compact interval $[0, T]$, $R=(-\infty, \infty)$ and let $f: I\times R^{nm}\to R^m$ be continuous. We consider the differential system

$$x^{(n)} + f(t, x, x', \cdots, x^{(n-1)}) = 0, \tag{1.1}$$

and give conditions which ensure that (1.1) has periodic solutions of small period, i.e., conditions which ensure the existence of a constant $\omega_0$, $0<\omega_0\leqq T$, such that for every $\omega$, $0<\omega\leqq\omega_0$, (1.1) has a solution $x(t)$ such that

$$x^{(i)}(0) = x^{(i)}(\omega), \qquad i = 0, \cdots, n-1. \tag{1.2}$$

The considerations in this paper are largely motivated by a paper of Seifert [7] where degree-theoretic arguments are used to prove the existence of periodic solutions (of small period) for the undamped oscillator

$$x'' + g(x) = p(t), \tag{1.3}$$

where $g(x)$ is a general restoring force having the property that

$$xg(x) > 0, \qquad x \neq 0, \qquad |g(x)| \to \infty \quad \text{as } |x| \to \infty. \tag{1.4}$$

Using only simple results from ordinary differential equations and from degree theory (we use the theory as developed in [**2**] and [**6**]) we establish the following general principle.

THEOREM 1. *Let there exist a nonempty bounded open set $A\subset R^m$ such that $f(0, x, 0, \cdots, 0)\neq 0$ for $x \in \partial A$ and let* $\deg(f(0, x, 0, \cdots, 0), A, 0)\neq 0$. *Then there exists $\omega_0$, $0<\omega_0\leqq T$, such that for every $\omega$, $0<\omega\leqq\omega_0$,* (1.1) *has a solution satisfying the periodic boundary conditions* (1.2).

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Received by the editors January 17, 1972 and, in revised form, April 17, 1972.
*AMS* 1970 *subject classifications*. Primary 34C25; Secondary 34A40, 34B15.
*Key words and phrases*. Periodic solutions, degree theory.
[1] Research supported in part by the U.S. Army grant ARO-D-31-124-71-G146.

Throughout the paper, we shall say that a differential equation has periodic solutions of small periods whenever the conclusion of Theorem 1 holds for that equation. Furthermore, we assume throughout that all solutions of the equation being considered exist on the basic interval $[0, T]$, and $\|\cdot\|$ shall denote the Euclidean norm in $R^m$. No ambiguity will arise if we use the symbol 0 for the zero of all Euclidean spaces considered in this paper.

## 2. Corollaries.

COROLLARY 1. *Let $f: I \times R^{nm} \to R^m$ be continuous and let there exist a constant $r > 0$ such that either*

(i) $x \cdot f(0, x, 0, \cdots, 0) > 0$, $\|x\| = r$, *or*

(ii) $x \cdot f(0, x, 0, \cdots, 0) < 0$, $\|x\| = r$.

*Then* (1.1) *has periodic solutions of small periods.*

PROOF. Let $A = \{x \in R^m : \|x\| < r\}$. Then either condition (i) or (ii) above implies that $\deg(f(0, x, 0, \cdots, 0), A, 0)$ is defined. Further, either of the conditions implies that the vector field $f(0, x, 0, \cdots, 0)$, $x \in \partial A$, has the property that $f(0, x, 0, \cdots, 0)$ and $f(0, -x, 0, \cdots, 0)$ do not have the same direction for every $x \in \partial A$ and hence that $f(0, x, 0, \cdots, 0)$ is homotopic to an odd vector field. It follows from the homotopy invariance theorem of degree theory (see **[2]** or **[6]**) and from Borsuk's theorem (see **[6]**) that $\deg(f(0, x, 0, \cdots, 0), A, 0) \neq 0$. We may, therefore, apply Theorem 1.

COROLLARY 2. *Consider the differential equation*

$$x^{(n)} + h(t, x, x', \cdots, x^{(n-1)}) + g(x) = p(t), \tag{2.1}$$

*where $h: I \times R^n \to R$, $g: R \to R$, $p: I \to R$ are continuous and have the property that $h(0, x, 0, \cdots, 0) = 0$, $xg(x) > 0$ $(< 0)$ and $|g(x)| \to \infty$ as $|x| \to \infty$. Then* (2.1) *has periodic solutions of small periods.*

PROOF. Since $p$ is continuous, $xg(x) > 0$ $(< 0)$ and $|g(x)| \to \infty$ as $|x| \to \infty$, there will exist a constant $r > 0$ such that either

(i) $x(g(x) - p(0)) > 0$, $|x| = r$, or

(ii) $x(g(x) - p(0)) < 0$, $|x| = r$.

We may, therefore, apply the previous corollary to complete the proof.

REMARK. Taking $h \equiv 0$, $n = 2$, we obtain the result of Seifert **[7]**. Also, if $n = 2$ and $h = k(x, x')$ we obtain the existence of periodic solutions of small period for the forced Lienard equation $x'' + k(x, x') + g(x) = p(t)$.

REMARK. In the case of equation (2.1), it will be apparent from the proof of Theorem 1 that the constant $\omega_0$ depends only on the left side of the equation and a bound on $|p(t)|$, $0 \leqq t \leqq T$. Thus if $p(t)$ is a periodic function (in the usual sense) of period $\omega$, $\omega \leqq \omega_0$, then the corresponding periodic solution (whose existence is guaranteed by Theorem 1) may be extended periodically so as to yield a periodic solution (in the usual sense) of (2.1). A similar remark also holds for equation (1.1).

COROLLARY 3. *Consider the nth order scalar equation*

$$x^{(n)} + f(t, x, x', \cdots, x^{(n-1)}) = 0, \tag{2.2}$$

*where* $f: I \times R^n \to R$ *is continuous, and let there exist constants* $\alpha$, $\beta$, $\alpha < \beta$, *such that either*

(i) $f(t, \alpha, 0, \cdots, 0) \leqq 0 \leqq f(t, \beta, 0, \cdots, 0)$, *or*

(ii) $f(t, \alpha, 0, \cdots, 0) \geqq 0 \geqq f(t, \beta, 0, \cdots, 0)$,

*with strict inequalities holding in either case for* $t=0$. *Then* (2.2) *has periodic solutions of small periods.*

PROOF. In either case, $\deg(f(0, x, 0, \cdots, 0), (\alpha, \beta), 0) \neq 0$.

REMARK. Corollary 3 represents generalizations of some results in [**4**] and [**5**]. Case (i) extends Theorem 1 of [**5**] in the sense that no local Lipschitz condition is required and further that $n$ need not equal 2. On the other hand, we need to assume here that strict inequalities hold for $t=0$ in order for $\deg(f(0, x, 0, \cdots, 0), (\alpha, \beta), 0)$ to be defined. Case (ii) extends some special cases of results in [**4**] to higher order equations; however, the results in [**4**] are valid for arbitrary periods whereas Corollary 3 only guarantees the existence of solutions of small periods.

3. **Proof of Theorem 1.** Before proceeding with the proof, we need some terminology and some preliminary lemmas.

Let $y=(x, x', \cdots, x^{(n-1)})$, $F(t, y)=(-x', \cdots, -x^{(n-1)}, f(t, y))$, $k=nm$, and consider the equivalent system of differential equations

$$y' + F(t, y) = 0. \tag{3.1}$$

A point $y_0 \in R^k$ is called an $\omega_0$-*nonrecurrence* point of (3.1) if every solution $y(t)$ of (3.1) with $y(0)=y_0$ is such that $y(t) \neq y_0$, $0 < t \leqq \omega_0$.

LEMMA 1. *Let* $\Omega \subset R^k$ *be a nonempty bounded open region whose boundary* $\partial\Omega$ *consists of* $\omega_0$-*nonrecurrence points only*, $\omega_0 \leqq T$. *Further, let* $\deg(F(0, y), \Omega, 0)$ *be defined and nonzero. Then for every* $\omega$, $0 < \omega \leqq \omega_0$, *there exists a solution* $y(t)$ *of* (3.1) *such that* $y(0)=y(\omega)$.

PROOF. This follows from the results of Krasnosel'skiĭ [**3**, pp. 79–83] and the observation that $\deg(F(0, y), \Omega, 0) \neq 0$ implies the nonvanishing of the rotation of the vector field considered by Krasnosel'skiĭ.

In the sequence of lemmas to follow, we shall show that the hypotheses of Theorem 1 allow us to construct a region $\Omega$ and find a number $\omega_0$ so that Lemma 1 may be applied to equation (3.1).

LEMMA 2. *Let $N>0$ be given and let $B\subset R^m$ be a bounded open set such that $\bar{A}\subset B$. Then there exists $\omega_1>0$, $0<\omega_1\leqq T$, such that every solution $x(t)$ of* (1.1) *with $x^{(i)}(0)=r_i$, $i=0,\cdots,n-1$, $r_0\in\bar{A}$, $\|r_i\|\leqq N$, $i=1,\cdots,n-1$, has the property that $x(t)\in\bar{B}$, $\|x^{(i)}(t)\|\leqq 2N$, $i=1,\cdots,n-1$, $0\leqq t\leqq\omega_1$.*

PROOF. This is an immediate consequence of the continuity of $f$ and the basic initial value problem existence results.

For $N$ and $B$ as defined above we let

$$\Omega=\{(x,x',\cdots,x^{(n-1)}):x\in A,\ \|x^{(i)}\|<N,\ i=1,\cdots,n-1\},$$

(3.2)

$$\Sigma=\{(x,x',\cdots,x^{(n-1)}):x\in B,\ \|x^{(i)}\|<2N,\ i=1,\cdots,n-1\},$$

and for every $K\subset\bar{\Omega}$ we let $S(K)$ denote the set of all solutions $y$ of (3.1) with $y(0)\in K$. Lemma 2 thus implies that if $y\in S(K)$ then $y(t)\in\bar{\Sigma}$ for $0\leqq t\leqq\omega_1$.

LEMMA 3. *For every $K\subset\bar{\Omega}$, $S(K)$ has compact closure in $C([0,\omega_1],R^k)$.*

PROOF. An application of the Ascoli-Arzela theorem.

LEMMA 4. *There exists $\omega_0$, $0<\omega_0\leqq\omega_1$, such that $\partial\Omega$ consists only of $\omega_0$-nonrecurrence points of* (3.1).

PROOF. Assume the contrary. Then there is a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n\to 0$ as $n\to\infty$, and a sequence of points $y_n\in\partial\Omega$ such that (3.1) has a solution $y_n(t)$ with $y_n(0)=y_n=y_n(t_n)$. By Lemma 2, $y_n(t)\in\bar{\Sigma}$, $0\leqq t\leqq\omega_1$. By passing to subsequences, if necessary, relabeling, and applying Lemma 3, we may assume that $\lim_{n\to\infty}y_n=y^*\in\partial\Omega$ and that $\lim_{n\to\infty}y_n(t)=y(t)$ is a solution of (3.1) with $y(0)=y^*$. Since $y_n(t)=y_n-\int_0^t F(s,y_n(s))\,ds$ we conclude that

$$\frac{1}{t_n}\int_0^{t_n}F(s,y_n(s))\,ds=0.$$

By continuity of $F$ we have

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\|F(s,y_n(s))-F(0,y^*)\|\,ds=0,$$

and therefore that $F(0,y^*)=0$.

Let $y^*=(x,x',\cdots,x^{(n-1)})$, then

$$F(0,y^*)=0=(-x',\cdots,-x^{(n-1)},f(0,y^*)),$$

which implies that $f(0,x,0,\cdots,0)=0$ for some $x\in\partial A$, a contradiction.

PROOF OF THEOREM 1. We now apply Lemma 1 with $\omega_0$ as chosen in Lemma 4 and $\Omega$ as given by (3.2). We only need to verify that $\deg(F(0,y),\Omega,0)\neq 0$.

Since

$$\deg(F(0,y),\Omega,0) = \deg(-x',\cdots,-x^{(n-1)},f(0,x,x',\cdots,x^{(n-1)}),\Omega,0),$$

and since $f(0,x,0,\cdots,0)\neq 0$ on $\partial A$, we conclude that $\deg(F(0,y),\Omega,0)$ is defined. Further $(-x',-x'',\cdots,-x^{(n-1)},f(0,x,\lambda x',\cdots,\lambda x^{(n-1)}))\neq 0$ on $\partial\Omega$, $0\leqq\lambda\leqq 1$; hence $F(0,y)$ is homotopic to $(-x',\cdots,-x^{(n-1)}, f(0,x,0,\cdots,0))$. By the homotopy invariance theorem of degree theory (see [2] or [6])

$$\deg(F(0,y),\Omega,0) = \deg(-x',\cdots,-x^{(n-1)},f(0,x,0,\cdots,0),\Omega,0).$$

The latter, on the other hand, is nonzero, if and only if

$$\deg(f(0,x,0,\cdots,0),A,0)\neq 0$$

(see [6]). This completes the proof.

REMARK. We note from the above lemmas and proofs that $\omega_0$ depends on the arbitrarily chosen constant $N$ and region $B$. Thus in varying both of these quantities one may possibly increase $\omega_0$ and hence increase the range of possible periods for solutions of (1.1) satisfying the periodic boundary conditions (1.2). The interested reader is referred to the paper [1] where existence results for periodic solutions (of period $T$) of systems of second order equations are established using methods similar to the ones used in this paper.

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