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THE p-CLASSES OF A HILBERT MODULE

JAMES F. SMITH

ABSTRACT. Let H be a right Hilbert module over a proper H^*-algebra A. For $0<p\leqq\infty$, an extended-real value $\|f\|_p$ is associated with each $f\in H$, and the p-class H_p is defined to be $\{f\in H:\|f\|_p<\infty\}$. For $1\leqq p\leqq\infty$, $(H_p, \|\cdot\|_p)$ is a right normed A-module. If $1\leqq p\leqq 2$, there is a conjugate-linear isometry of $(H_p, \|\cdot\|_p)$ onto the dual of $(H_q, \|\cdot\|_q)$, where $(1/p)+(1/q)=1$; hence H_p is complete in its norm.

1. **Introduction.** Let A be a proper H^*-algebra with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. By a projection in A we mean a nonzero selfadjoint idempotent, and by a projection base for A we mean a maximal family of mutually orthogonal projections. The trace class of A, denoted by τA, is the set $\{xy:x, y\in A\}$. It is shown in [6] that a trace functional tr is unambiguously defined on τA by letting $\operatorname{tr} xy=\langle x, y^*\rangle=\sum\langle xyp_\omega, p_\omega\rangle$, where $\{p_\omega:\omega\in\Omega\}$ is any projection base for A. It is further shown that for each nonzero $a\in A$ there exists a unique positive element $[a]\in A$ (that is, one possessing the property $\langle[a]x, x\rangle\geqq 0$ for every $x\in A$) such that $[a]^2=a^*a$; moreover, $a\in\tau A$ if and only if $[a]\in\tau A$. A norm τ is defined on τA by letting $\tau(a)=\operatorname{tr}[a]$; then $(\tau A, \tau)$ is a Banach $*$-algebra ([6], [5]). In [7] the present author has shown that each nonzero positive element b of A has a unique spectral representation $b=\sum\lambda_n e_n$, where the λ_i are positive numbers with $\lambda_i<\lambda_j$ if $i>j$, and the e_i are mutually orthogonal projections. In particular, for any nonzero $a\in A$, if $\sum\lambda_n e_n$ is the spectral representation of $[a]$, we define $|a|_p$, for $0<p<\infty$, by $|a|_p^p=\sum\lambda_n^p|e_n|^2$. We also define $|a|_\infty$ to be λ_1, and $|0|_p=0$ for $0<p\leqq\infty$. The p-class A_p, $0<p\leqq\infty$, is then defined as $\{a\in A:|a|_p<\infty\}$. Among the results of [7] are the following: (1) $|a|_\infty=\|L_a\|$, where L_a denotes, as usual, the left multiplication operator; (2) $A_p\subset A_{p'}$, if $0<p<p'\leqq 2$, the inclusion being proper if A is infinite-dimensional; and $A_p=A$ if $p\geqq 2$; (3) $(A_2, |\cdot|_2)=(A, |\cdot|)$ and $(A_1, |\cdot|_1)=(\tau A, \tau)$; (4) $(A_p, |\cdot|_p)$ is a normed $*$-algebra for $1\leqq p\leqq\infty$, and is complete for $1\leqq p\leqq 2$.

A (right) Hilbert A-module H, introduced by Saworotnow in [4], is a complex linear space which is a right module over the proper H^*-algebra

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A, and on which there is defined a vector inner product $(\cdot, \cdot)$ mapping $H\times H$ into τA, such that for elements of H: (1) $(f+g, h)=(f, h)+(g, h)$; (2) $(f, g)=(g, f)^*$; (3) $(f, ga)=(f, g)a$, where $a\in A$; (4) $(f, \alpha g)=\alpha(f, g)$ for any complex number α; (5) if $f\neq 0$ then $(f, f)=a^2$ for some (unique) positive element $a\neq 0$ in A; we denote this a by $[f]$; (6) H is complete in the norm $\|\cdot\|$ derived from the inner product $[\cdot, \cdot]$ defined by $[f, g]=\operatorname{tr}(g, f)$. Basic properties of Hilbert A-modules are obtained in [**4**], among them the fact that $\|fa\|\leqq\|f\|\ |a|$ for $f\in H$, $a\in A$; hence a Hilbert A-module is evidently a particular instance of a Banach module (see [**2**, p. 263]). (We shall assume without loss of generality that H is a faithful module, since its right annihilator R is a closed two-sided ideal of A; hence H is always a faithful Hilbert $R^\perp$-module.) Giellis [**1**] has defined the trace class of H to be $\tau H=\{fa: f\in H, a\in A\}$ and has defined a norm π on τH by $\pi(f)=\tau([f])$. He has shown that $(\tau H, \pi)$ is a Banach module and has presented results on duality relationships.

Our present aim is to generalize the results of [**1**] for Hilbert modules, much as those of [**5**] and [**6**] were generalized for H^*-algebras in [**7**]. For $f\in H$, and for $0<p\leqq\infty$, we define $\|f\|_p=|[f]|_p$, and we let $H_p=\{f\in H: \|f\|_p<\infty\}$ (that is, $f\in H_p$ if and only if $[f]\in A_p$). Our results are the following.

THEOREM 1. *For $0<p<p'\leqq\infty$ and any $f\in H$, $\|f\|_{p'}\leqq\|f\|_p$; hence $H_p\subset H_{p'}$, and $H_p=H$ if $p\geqq 2$. For $1\leqq p\leqq\infty$, $(H_p, \|\cdot\|_p)$ is a right normed A-module. $(H_2, \|\cdot\|_2)=(H, \|\cdot\|)$, and $(H_1, \|\cdot\|_1)=(\tau H, \pi)$.*

THEOREM 2. *For $1\leqq p\leqq 2$, let q be such that $(1/p)+(1/q)=1$. Then there exists a conjugate-linear isometry of $(H_p, \|\cdot\|_p)$ onto the dual of $(H_q, \|\cdot\|_q)$; hence H_p is complete in its norm.*

We conclude with a necessary and sufficient condition for the inclusion $H_p\subset H_{p'}$ to be proper $(1\leqq p<p'\leqq 2)$.

2. **Preliminary results.** We recall first of all from [**4**, Lemma 1] that $(fa, g)=a^*(f, g)$ for any $f, g\in H$ and $a\in A$. From the fact that $\tau(f, g)\leqq \|f\|\ \|g\|$ [**4**, Theorem 2], it is easily established that $(\cdot, \cdot)$ is τ-continuous on $H\times H$ and therefore $|\cdot|$-continuous as well, since τ dominates $|\cdot|$ [**6**, Corollary 3]. We observe that $[f, ga]=\operatorname{tr}(ga, f)=\operatorname{tr} a^*(g, f)=\langle(g, f), a\rangle$; similarly, $[fa, g]=\langle a, (f, g)\rangle$. Also, $[f, ga]=\operatorname{tr}(g, f)a^*=\operatorname{tr}(g, fa^*)=[fa^*, g]$.

As in [**1**], we define the following sets of bounded linear transformations:

$$R(A)=\{T: A\to A \mid T(ab)=(Ta)b \text{ for all } a, b\in A\},$$
$$R(AH)=\{T: A\to H \mid T(ab)=(Ta)b \text{ for all } a, b\in A\},$$
$$R(HA)=\{T: H\to A \mid T(fa)=(Tf)a \text{ for all } f\in H, a\in A\}.$$

Still following [**1**], we define $L_f \in R(HA)$ for each $f \in H$ by $L_f g = (f, g)$. (Note that, by standard notation, $L_{[f]}$ is the operator effecting left multiplication by $[f]$ in A; clearly, $L_{[f]} \in R(A)$.) For $f \in H$, we shall define $T_f \in R(AH)$ by $T_f a = fa$. By a remark of Giellis [**1**, p. 65], we have $T_f^* T_f = L_{[f]}^2 = L_{[f]^2}$. The relationship $T_f = L_f^*$ also holds, since for any $f, g \in H$ and $a \in A$, $[g, T_f a] = [g, fa] = \langle (f, g), a\rangle = \langle L_f g, a\rangle = [g, L_f^* a]$.

LEMMA 1. *For any $f \in H$ and $a \in A$, $\|fa\| = |[f]a|$.*

PROOF. $\|fa\|^2 = [T_f a,\ T_f a] = \langle a,\ T_f^* T_f a\rangle = \langle a,\ [f]^2 a\rangle = \langle [f]a,\ [f]a\rangle = |[f]a|^2$.

COROLLARY 1. $\|f\|_\infty = |[f]|_\infty = \|L_{[f]}\| = \|T_f\| = \|L_f\|$.

For any $f \neq 0$ in H, let $\sum \lambda_n e_n$ be the spectral representation of $[f]$ [7, Theorem 2.5]. We shall denote the countable (possibly finite) set $\{e_n\}$ by $E_{[f]}$ and refer to it as the spectral family of $[f]$. Any projection base $\{e_\omega : \omega \in \Omega\}$ containing every $e_n \in E_{[f]}$ will be called a projection base associated with $[f]$. In [**1**, Lemma 1] it is shown that if $\sum \lambda_n e_n$ is the spectral representation of $[f]$, then the operator $W_f \in R(AH)$ defined for any $x \in A$ by $W_f x = \sum \lambda_n^{-1} f e_n x$ is a partial isometry with $f = W_f[f]$ and $[f] = W_f^* f$. We shall refer to W_f as the partial isometry associated with f.

The proofs of our next two lemmas make use of the fact that for any $S \in R(A)$ and $1 \leqq p \leqq \infty$ we have $|Sa|_p \leqq \|S\|\ |a|_p$, where $\|S\|$ denotes the norm of S as an operator on $(A, |\cdot|)$ [7, Proposition 3.19]. Lemma 2 gives a similar result for $T \in R(HA)$.

LEMMA 2. *If $T \in R(HA)$, then $|Tf|_p \leqq \|T\|\ \|f\|_p$ $(1 \leqq p \leqq \infty)$.*

PROOF. $|Tf|_p = |TW_f[f]|_p \leqq \|TW_f\|\ |[f]|_p \leqq \|T\|\ \|W_f\|\ \|f\|_p \leqq \|T\|\ \|f\|_p$.

LEMMA 3. *If $a \in A$ and $T \in R(HA)$, then there exists $g \in H$ such that $L_a T = L_g$; moreover, $\|g\|_p \leqq \|T\|\ |a|_p$ $(1 \leqq p \leqq \infty)$, where $\|T\|$ is the norm of T as a transformation from $(H, \|\cdot\|)$ to $(A, |\cdot|)$.*

PROOF. The first part of the lemma is Lemma 7 of [**1**]. We shall show that $\|g\|_p \leqq \|T\|\ |a|_p$, noting that this result is obvious for $p = \infty$, in view of Corollary 1. Assume now that $1 \leqq p < \infty$ and that $\sum \lambda_n e_n$ is the spectral representation of $[g] \neq 0$. As shown above, we have $T_g = L_g^* = T^* L_a^* = T^* L_{a^*}$, so that $T_g[g] = g[g] = T^* a^*[g]$, and therefore

$$W_g^* T^* a^*[g] = W_g^* g[g] = [g]^2 = \sum \lambda_n^2 e_n.$$

It follows that $W_g^* T^* a^*[g] e_n = \lambda_n^2 e_n = \lambda_n W_g^* T^* a^* e_n$, and we conclude that $W_g^* T^* a^* e_n = \lambda_n e_n = e_n (W_g^* T^* a^*)^*$. Now for any $b \in A$, let $P_k b = \sum_{n=1}^k e_n b$.

P_k is the orthogonal projection onto the right ideal $\sum_{n=1}^{k} e_n A$ in A, and $P_k \in R(A)$. We have

$$P_k(W_g^* T^* a^*)^* = \sum_{n=1}^{k} e_n (W_g^* T^* a^*)^* = \sum_{n=1}^{k} \lambda_n e_n.$$

Thus, for each k,

$$\begin{aligned}\sum_{n=1}^{k} \lambda_n^p |e_n|^2 &= |P_k(W_g^* T^* a^*)^*|_p^p \leqq \|P_k\|^p \, |W_g^* T^* a^*|_p^p \\ &\leqq \|W_g^* T^*\|^p \, |a^*|_p^p \leqq \|W_g^*\|^p \, \|T^*\|^p \, |a|_p^p \leqq \|T\|^p \, |a|_p^p.\end{aligned}$$

(We have used the fact that $|a|_p = |a^*|_p$ [7, Corollary 3.16].) Therefore, $|[g]|_p^p \leqq \|T\|^p |a|_p^p$, or $\|g\|_p \leqq \|T\| \, |a|_p$.

3. **Proof of Theorem 1.** The first statement of the theorem is evident from the corresponding statements about A_p [7, Corollary 3.12]. To show that H_p $(1 \leqq p \leqq \infty)$ is a linear space we verify that $\|\cdot\|_p$ is a linear space norm, a fact which is obvious for $p = \infty$, by Corollary 1. To establish subadditivity for $1 \leqq p < \infty$, let f and g be any elements of H_p, and let W be the partial isometry associated with $f+g$. We then have

$$\begin{aligned}\|f + g\|_p &= |W^*(f + g)|_p = |W^* f + W^* g|_p \\ &= |W^* W_f[f] + W^* W_g[g]|_p \leqq |W^* W_f[f]|_p + |W^* W_g[g]|_p \\ &\leqq \|W^* W_f\| \, |[f]|_p + \|W^* W_g\| \, |[g]|_p \leqq \|f\|_p + \|g\|_p.\end{aligned}$$

We have again used Proposition 3.19 of [7], as well as the triangle inequality for $|\cdot|_p$ [7, Proposition 3.23]. The remaining properties of a linear space norm are readily verified. Now for any $f \in H$, $a \in A$, let W be the partial isometry associated with fa. We have $\|fa\|_p = |[fa]|_p = |W^* fa|_p \leqq |W^* f|_p |a|_\infty$ by [7, Corollary 3.20]. Since $|a|_\infty \leqq |a|$ [7, Lemma 3.9],

$$|W^* f|_p \, |a|_\infty \leqq \|W^*\| \, \|f\|_p \, |a| \leqq \|f\|_p \, |a|.$$

Thus H_p is a right normed A-module. The final statement of the theorem follows from corresponding results in A [7, Remark 3.5], along with [**1**, Lemma 2]: for any $f \in H$,

$$\|f\|_2^2 = |[f]|_2^2 = |[f]|^2 = \operatorname{tr}[f]^2 = \operatorname{tr}(f, f) = [f, f] = \|f\|^2,$$

and

$$\|f\|_1 = |[f]|_1 = \tau([f]) = \pi(f).$$

We remark that the completeness of $(H_p, \|\cdot\|_p)$ for $1 \leqq p < 2$ can be established by the method of [**3**, p. 265] as adapted in [**1**]; however we omit this proof since it is rendered unnecessary by Theorem 2.

4. **Proof of Theorem 2.** The case $p=1$ is Theorem 2 of [**1**]. For $1<p\leqq 2$ we observe that $2\leqq q<\infty$. For any $g\in H_q$ $(=H)$ and any $f\in H_p$, let $\phi_f(g)=[g,f]=\operatorname{tr}(f,g)$. Clearly, ϕ_f is a linear functional on H_q and the mapping $f\to\phi_f$ is conjugate-linear. We shall show first that ϕ_f is bounded and that $\|\phi_f\|\leqq\|f\|_p$. If $E=\{e_\omega:\omega\in\Omega\}$ is any projection base for A, we have

$$|\phi_f(g)| = |\operatorname{tr}(f,g)| = |\operatorname{tr}(g,f)| = \left|\sum\langle(g,f)e_\omega, e_\omega\rangle\right| \leqq \sum|\langle(g,fe_\omega), e_\omega\rangle|$$
$$= \sum|[fe_\omega, ge_\omega]| \leqq \sum\|fe_\omega\|\,\|ge_\omega\| = \sum|[f]e_\omega|\,|[g]e_\omega|,$$

by Lemma 1. If E is now taken to be a projection base associated with $[f]$, we conclude as in the proof of [7, Lemma 3.25] that this last sum does not exceed $|[f]|_p|[g]|_q=\|f\|_p\|g\|_q$.

To show that $\|f\|_p\leqq\|\phi_f\|$, we consider the linear functional $\theta_{[f]}$ defined on A_q by $\theta_{[f]}(a)=\operatorname{tr} a[f]$. From [7, Proposition 3.26], we have $\|\theta_{[f]}\|=|[f]|_p=\|f\|_p$; hence it suffices to show that $\|\theta_{[f]}\|\leqq\|\phi_f\|$. Let a be any element of A_q. Then

$$|\theta_{[f]}(a)| = |\operatorname{tr} a[f]| = |\operatorname{tr} aW_f^*f| = |\operatorname{tr} L_g f|$$
$$= |\operatorname{tr}(g,f)| = |\operatorname{tr}(f,g)| = |\phi_f(g)|,$$

where, by Lemma 3, $g\in H$ is such that $L_aW_f^*=L_g$, with $\|g\|_q\leqq|a|_q$. Using this last inequality we conclude that $\|\theta_{[f]}\|\leqq\|\phi_f\|$.

We must show, finally, that the mapping $f\to\phi_f$ is *onto* the dual of H_q. Let ϕ be any bounded linear functional on H_q. For each $g\in H_q$ $(=H)$, $|\phi(g)|\leqq\|\phi\|\,\|g\|_q\leqq\|\phi\|\,\|g\|$, since $q\geqq 2$ .Thus ϕ is bounded on $(H,\|\cdot\|)$ and there exists $f\in H$ such that $\phi(g)=[g,f]$. We need only show that $f\in H_p$. Let $\sum\lambda_n e_n$ be the spectral representation of $[f]$, and let

$$v_k = \sum_{n=1}^{k}\lambda_n^{p-1}e_n\in A_q;$$

then

$$|v_k|_q = \left(\sum_{n=1}^{k}\lambda_n^{pq-q}|e_n|^2\right)^{1/q} = \left(\sum_{n=1}^{k}\lambda_n^{p}|e_n|^2\right)^{1/q}.$$

Using Lemma 3, we take $g_k\in H$ such that $L_{g_k}=L_{v_k}W_f^*$, where $\|g_k\|_q\leqq|v_k|_q$. Then for each k,

$$\sum_{n=1}^{k}\lambda_n^p|e_n|^2 = \left|\operatorname{tr}\sum_{n=1}^{k}\lambda_n^p e_n\right| = \left|\operatorname{tr}\left(\sum_{n=1}^{k}\lambda_n^{p-1}e_n\right)[f]\right| = |\operatorname{tr} v_k[f]|$$
$$= |\operatorname{tr} v_kW_f^*f| = |\operatorname{tr}(g_k,f)| = |[g_k,f]| = |\phi(g_k)|$$
$$\leqq \|\phi\|\,\|g_k\|_q \leqq \|\phi\|\,|v_k|_q = \|\phi\|\left(\sum_{n=1}^{k}\lambda_n^p|e_n|^2\right)^{1/q}.$$

Thus $(\sum_{n=1}^{k} \lambda_n^p |e_n|^2)^{1/p} \leqq \|\phi\|$ for each k, and consequently $\|f\|_p \leqq \|\phi\| < \infty$.

5. **Conditions for distinctness of the p-classes.** Suppose $1 \leqq p < p' \leqq 2$. In the H^*-algebra A, for A_p to be a proper subset of $A_{p'}$ it is necessary and sufficient that A be infinite-dimensional [7, Proposition 3.14]. We shall give a condition for the corresponding relationship to hold in the case of H_p and $H_{p'}$.

An element of H will be called primitive if it is of the form $fe \neq 0$, where e is a primitive projection in A; if $\|fe\| = |e|$, fe will be called a normal primitive element. (Note that the primitive projection e is uniquely determined for fe, since if $fe = gp$, where p is a primitive projection in A, then $e(f, f)e = p(g, g)p = \alpha e = \beta p \neq 0$; hence $e = p$.) A pair of primitive elements $f_1 e_1$ and $f_2 e_2$ will be called doubly orthogonal if $(f_1 e_1, f_2 e_2) = 0$ and $\langle e_1, e_2 \rangle = 0$.

PROPOSITION 1. *For $1 \leqq p < p' \leqq 2$, H_p is a proper subset of $H_{p'}$ if and only if H contains an infinite set of pairwise doubly orthogonal primitive elements.*

PROOF. We note first that there exist nonempty sets of pairwise doubly orthogonal primitive elements in H, since for any $f \in H$ there is a primitive projection e such that $fe \neq 0$ (Lemma 1). Now suppose that every maximal set of such elements is finite, and let $\{f_1 e_1, \cdots, f_k e_k\}$ be a maximal set. We may assume that the $f_n e_n$ are normal. We have $[f_n e_n]^2 = (f_n e_n, f_n e_n) = e_n [f_n e_n]^2 e_n = \alpha_n^2 e_n$ for positive α_n, since $[f_n e_n]^2$ is a positive element of A. Hence $[f_n e_n] = \alpha_n e_n$, and from $\alpha_n |e_n| = |[f_n e_n]| = \|f_n e_n\| = |e_n|$ we conclude that $\alpha_n = 1$. Thus for any $a \in H$, $\|f_n e_n a\| = |[f_n e_n] e_n a| = |e_n a|$, and hence $f_n e_n A$ is a closed submodule of A isomorphic to the closed right ideal $e_n A$. Let $M = \sum_{n=1}^{k} f_n e_n A$. Then $H = M \ominus M^p$, where $M^p = \{f \in H: (f, g) = 0 \text{ for all } g \in M\}$ [4, corollary to Lemma 3]. Clearly, every element of M belongs to the trace class τH. We shall show that the same is true for elements of M^p. Let $\{e_\omega : \omega \in \Omega\}$ be a projection base for A containing $\{e_1, \cdots, e_k\}$. For any $f \in M^p$ and any $e_\alpha \neq e_n$ $(n = 1, \cdots, k)$, $fe_\alpha = 0$ or else fe_α would be a primitive element doubly orthogonal to each $f_n e_n$ $(n = 1, \cdots, k)$, contradicting maximality. Thus $[f] e_\alpha = 0$, by Lemma 1, and we have $[f] = \sum [f] e_\omega = \sum_{n=1}^{k} [f] e_n$; hence $[f] \in \tau A$ and $f \in \tau H$. Since H is identical with its trace class, $H_p = H$ for $1 \leqq p \leqq 2$.

Suppose, to the contrary, that H contains an infinite set $\{f_n e_n : n \in N\}$ of pairwise doubly orthogonal (normal) primitive elements. For $1 \leqq p < p' \leqq 2$, choose r with $p < r < p'$, and consider the series $\sum n^{-1/r} |e_n|^{-2/p'} f_n e_n$. The terms of this series are mutually orthogonal in $(H, \|\cdot\|)$; and, recalling from above that $[f_n e_n] = e_n$, we easily show that the squares of their norms have a finite sum. Thus there exists $f \in H$ such that $f = \sum n^{-1/r} |e_n|^{-2/p'} f_n e_n$.

Now by the continuity of $(\cdot,\cdot)$ on $H\times H$, we have $[f]^2=(f,f)=\sum n^{-2/r}|e_n|^{-4/p'}(f_ne_n, f_ne_n)=\sum n^{-2/r}|e_n|^{-4/p'}e_n$; and therefore $[f]=\sum n^{-1/r}|e_n|^{-2/p'}e_n$. It is now a simple matter to show, just as in [7, Proposition 3.14], that $f\in H_{p'}$, but $f\notin H_p$.

We close by remarking that for the condition of Proposition 1 to hold, A must necessarily be infinite-dimensional, as is evident. This is not sufficient, however, as is shown by the Hilbert A-module eA, where e is a primitive projection and A is topologically simple (the latter condition assuring that the module is faithful). However, by means of Theorem 6 of [4], along with the accompanying examples, it is readily possible to provide instances of Hilbert A-modules possessing the property of Proposition 1.

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Department of Mathematics, Le Moyne College, Syracuse, New York 13214