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SELECTION OF REPRESENTING MEASURES FOR INNER PARTS

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ABSTRACT. If a compact convex set K has an inner part Δ then there is a selection of pairwise boundedly absolutely continuous representing measures for Δ if and only if K is finite dimensional.

Let K denote a compact convex set in a LCTVS, $A(K)$ the affine continuous real functions on K, $\mathscr{P}(K)$ the set of regular Borel probability measures on K. Let $\Phi:\mathscr{P}(K)\to K$ be the map which associates to each measure μ its barycentre. Then Φ is affine, weak* continuous, and onto K. If $\Phi(\mu)=x$ we say μ *represents* x.

If L is any convex set, $x, y \in L$ and $r>0$, we say $[x, y]$ *extends by r in L* if $x+r(x-y) \in L$ and $y+r(y-x) \in L$. We write $x\sim y$ if $\exists\, r>0$ such that $[x, y]$ extends by r in L. This is an equivalence relation on L and the equivalence classes are the *parts* of L. It is easy to show that Φ carries parts into parts: If Π is a part of $\mathscr{P}(K)$ then $\Phi(\Pi)$ is contained in a part of K. Conversely if Δ is a part of K and F is any finite subset of Δ then there exists a part Π of $\mathscr{P}(K)$ such that $F\subset\Phi(\Pi)$. Indeed if $F=\{x_1, x_2, \cdots, x_n\}$ choose y_i and z_i in K such that $x_1 \in (y_i, z_i)$, the open line segment with endpoints y_i and z_i, and $x_i \in (y_i, x_1)$ $(2\leqq i\leqq n)$. If $\Phi(\mu_i)=y_i$ and $\Phi(\nu_i)=z_i$ for $\mu_i, \nu_i \in \mathscr{P}(K)$, then the part Π containing $\sum (\mu_i+\nu_i)/(2n-2)$ satisfies $F\subset\Phi(\Pi)$. Indeed since $x_1 \in (y_i, z_i)$ for each i, we can clearly find a representing measure ω for x_1 in Π. Since $x_i \in (y_i, x_1)$, an affine combination of μ_i and ω yields a representing measure for x_i in Π.

Thus if Δ is a part of K one might ask whether

$$(1) \qquad \Delta = \Phi(\Pi) \quad \text{for some part } \Pi \text{ of } \mathscr{P}(K).$$

Indeed Bear posed this question in [3] and reproduced an example of Har'kova [4] to show that (1) need not hold if $\mathscr{P}(K)$ is replaced by $\mathscr{P}(\Gamma)$ where Γ is the Shilov boundary of $A(K)$.

Since two probability measures μ and ν on K are in the same part of $\mathscr{P}(K)$ if and only if $\mu\leqq k\nu$ and $\nu\leqq k\mu$ for some k, condition (1) asserts

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the existence for Δ of a selection of representing measures on K which are pairwise boundedly absolutely continuous. There are two special cases when (1) is true for all parts Δ of K. One is when K is a simplex, for then there are unique maximal representing measures [**6**, §9], the other when K is finite dimensional (Theorem 1).

Let $K^i=\{x \in K: (\forall\, y \in K)(\exists\, r>0)x+r(x-y) \in K\}$. It can happen that $K^i=\varnothing$, but if $K^i \neq \varnothing$ it is a part of K called the *inner part*. Finite dimensional convex sets, for example, always have nonempty inner parts. In Theorem 1 we show that if $\Delta=K^i\neq\varnothing$ then (1) holds for Δ if and only if K is finite dimensional.

First some preliminaries. If L is a compact convex set, x, $y \in L$ and $x \sim y$; let

$$d(x, y) = \inf\{\log(1 + 1/r): [x, y] \text{ extends by } r\}.$$

In [**3**, Lemma 3.4] it is shown that d is a metric on each part of L, called the *part metric*. Now denote by d and D the part metrics on K and $\mathscr{P}(K)$ respectively and let

$$b(x, r) = \{y \in K: d(x, y) \leqq r\} \quad \text{and} \quad B(\mu, r) = \{\nu \in \mathscr{P}(K): D(\mu, \nu) \leqq r\}.$$

LEMMA 1. *Suppose Δ is a part of K, Π a part of $\mathscr{P}(K)$, and $\Delta=\Phi(\Pi)$. Then there exist $\mu \in \Pi$ and positive numbers M and k such that if $x=\Phi(\mu)$ then*

$$b(x, \log(1 + 1/M)) \subset \Phi(B(\mu, \log k)).$$

PROOF. If $\nu \in \Pi$ then the sets $\Phi(B(\nu, r))$ are closed in the part metric topology. Indeed suppose $x_n=\phi(\mu_n)$ with $\mu_n \in B(\nu, r)$ and $d(x_n, x)\to 0$. Choose a subset μ_{n_α} converging weak* to μ. Since $B(\nu, r)$ is weak* closed (easy to check), $\mu \in B(\nu, r)$. Since Φ is weak* continuous, x_{n_α} converges in K to $\Phi(\mu)$. But since $d(x_{n_\alpha}, x)\to 0$, x_{n_α} converges in K to x, hence $x=\Phi(\mu) \in \Phi(B(\nu, r))$. (It is an easily verified general fact that in any part of a compact convex set the part metric topology is stronger than the relativized compact topology.)

Since $\Delta=\Phi(\Pi)=\bigcup_{n=1}^{\infty} \Phi(B(\nu, n))$ and the part metric on Δ is complete [**1**, §3], the Baire category theorem tells us that we can find $x \in \Delta$ and integers h and M such that $b(x, \log(1+1/M))\subset \Phi(B(\nu, h))$. Choose $\mu \in \Pi$ such that $\Phi(\mu)=x$ and choose k such that $B(\nu, h)\subset B(\mu, \log k)$. $\square$

LEMMA 2. *Suppose $x \in K^i$. Then $\exists\ \delta>0$ such that*

$$y \in K \Rightarrow x + \delta(x - y) \in K.$$

PROOF. Let $H=K-x$. Then $0 \in H^i$ and so $H \cap -H$ is closed, convex and absorbs each point of H and $-H$. Since H is compact, convex,

$H \cap -H$ absorbs H [5, Corollary 10.2]. Thus $\exists\ \delta>0$ such that $\delta H \subset H \cap -H \subset -H$. Thus

$$y \in K \Rightarrow y - x \in H \Rightarrow \delta(y-x) \in -H \Rightarrow x + \delta(x-y) \in K. \quad \square$$

If A is a normed linear space and $\varepsilon \geqq 0$ let $B_\varepsilon = \{h \in A : \|h\| \leqq \varepsilon\}$.

LEMMA 3. *Suppose E is a normed linear space and G is a weak* closed subspace of the dual space E^*. Suppose $x \in E^*$, $r \geqq 0$ and $(x+B_r) \cap G = \varnothing$. Then $\exists\ f \in E$ such that $\|f\|=1$, $f(G)=0$ and $f(x)>r$.*

PROOF. $x+B_r$ is weak* compact and G is weak* closed. Hence $\exists\ f \in E$ such that $\|f\|=1$, $f(G)<\alpha$ and $f(x+B_r) \geqq \alpha$ for some α. Since G is a subspace, $\alpha>0$ and $f(G)=0$. Since $\|f\|=1$ we can find $y \in B_r$ such that $f(y)>r-\alpha$. Then $x-y \in x+B_r$ so $f(x-y) \geqq \alpha$ hence $f(x) \geqq \alpha + f(y) > r$. $\square$

Now for the main theorem. We always think of K as embedded in the Banach space $A(K)^*$ with the weak* topology. The norm of $A(K)^*$ provides a metric topology on K which we will refer to as the norm topology.

THEOREM 1. *Suppose $\Delta = K^i \neq \varnothing$. Then the following are equivalent.*

(1) $\Delta = \Phi(\Pi)$ *for some part* Π *of* $\mathscr{P}(K)$.

(2) K *is finite dimensional.*

PROOF. (1)$\Rightarrow$(2). Suppose (1) and suppose that K is metrizable. We will show that, in this case, K is finite dimensional. Then we will reduce the general case to this one.

We first show that K is norm separable. If $\mu \in \Pi$ then $\Pi \subset L^1(\mu)$ (via Radon Nikodym), and the norm topology that Π gets from $L^1(\mu)$ is the same as the norm topology it gets as a subset of $\mathscr{C}(K)^*$. Indeed if $g, h \in L^1(\mu)$ then

$$\sup_{f \in \mathscr{C}(K):\|f\|_\infty=1} \int f(g-h)\,d\mu = \|g\,d\mu - h\,d\mu\|$$
$$= \sup_{f \in L^\infty:\|f\|_\infty=1} \int f(g-h)\,d\mu = \|g-h\|_1,$$

where $\|\cdot\|$ denotes the variation norm in the Banach space $\mathscr{M}(K)$ of Radon measures on K. Since $L^1(\mu)$ is separable (K is metrizable), Π is norm separable in $\mathscr{C}(K)^*$. Since Φ is the restriction to $\mathscr{P}(K)$ of the natural, norm-decreasing surjection $\Phi : \mathscr{C}(K)^* \to A(K)^*$, $\Delta = \Phi(\Pi)$ is norm separable. Since $\Delta = K^i$ is norm dense in K, K is norm separable.

Now we show that K is norm compact. Since K is norm complete it will be enough to find for any $\varepsilon>0$ a finite set $F \subset A(K)^*$ such that $K \subset F + B_{2\varepsilon}$. So suppose $\varepsilon>0$. Choose μ, M, k and x from Lemma 1 and δ from

Lemma 2 so that $\delta(1+1/M) \leqq 1$. Since K is norm separable, we can cover K with countably many balls of norm radius $r=\varepsilon\delta/2M$. A finite number of these balls contains all but at most $\gamma=\varepsilon\delta/2kM$ of the measure μ. Let P be a finite dimensional subspace of $A(K)^*$ containing x and the centres of these finitely many balls.

We claim that $K \subset P+B_\varepsilon$. Indeed suppose $y \in K$ but $y \notin P+B_\varepsilon$. Let $z=x+(\delta/M)(y-x)$. Then $z \in K$ and $d(x, z) \leqq \log(1+1/M)$. Indeed $x+M(x-z)=x+\delta(x-y)$ which is in K by Lemma 2, and $z+M(z-x)=x+\delta(1+1/M)(y-x)$ which is in K since $\delta(1+1/M) \leqq 1$. So by Lemma 1 we can choose $\nu \in B(\mu, \log k)$ such that $z=\Phi(\nu)$. An easy computation shows that $d\nu=g\,d\mu$ with $1/k \leqq g \leqq k$. Also, since P is weak* closed and $y \notin P+B_\varepsilon$ we can find $f \in A(K)$ such that $\|f\|=1$, $f(P)=0$ and $f(y)>\varepsilon$ (Lemma 3). Then

$$f(z) = (\delta/M)f(y) > \varepsilon\delta/M, \quad \text{and}$$

$$\begin{aligned} \nu(f) &= \int fg\,d\mu = \int_{|f| \leqq r} fg\,d\mu + \int_{|f|>r} fg\,d\mu \\ &\leqq r\int g\,d\mu + \|f\|\,k \cdot \mu(\{|f|>r\}) \leqq r + k\gamma = \varepsilon\delta/M \end{aligned}$$

(where $\mu(\{|f|>r\}) \leqq \gamma$ since $|f(w)|>r \Rightarrow w \notin P+B_r$). Since $f \in A(K)$ and $\Phi(\nu)=z$ we must have $\nu(f)=f(z)$, a contradiction.

So $K \subset P+B_\varepsilon$. Hence $K \subset [(K+B_\varepsilon)\cap P]+B_\varepsilon$. Now $(K+B_\varepsilon)\cap P$ is finite dimensional and norm bounded, so relatively norm compact, and we can choose a finite set $F \subset A(K)^*$ so that $F+B_\varepsilon$ contains it. Hence $K \subset F+B_{2\varepsilon}$.

So K is norm compact. We deduce that the unit ball B_1 of $A(K)^*$ is norm compact. Indeed it follows from the Hahn Banach Theorem that every element of $A(K)^*$ is given by a Radon measure on K. Use the Hahn decomposition of this measure and the fact that any probability measure on K has a barycentre in K to deduce that, for any $\lambda \in B_1$, there exists $k, h \in K$ and $0 \leqq \alpha, \beta \leqq 1$ such that

$$\lambda(f) = \alpha f(k) - \beta f(h) \qquad (f \in A(K)).$$

Thus B_1 is contained in a continuous image of $K \times K \times [0, 1] \times [0, 1]$, and is norm compact. It follows that $A(K)^*$ is finite dimensional, and so is K.

Now drop the metrizability assumption; suppose K has (1) but is not finite dimensional. Choose a countably infinite, linearly independent sequence $\{f_n\} \subset A(K)$ such that $\|f_n\| \leqq 2^{-n}$. Define the map Ψ from K into l^2 by $\Psi(x)_n=f_n(x)$. Ψ is affine and continuous, hence maps K onto a compact convex subset H of l^2. From Lemma 4 below $H^i=\Psi(K^i)\neq\varnothing$. Since every $x \in K^i$ has a representing measure in Π, every $h \in H^i$ has a representing measure in $\Pi \circ \Psi^{-1} = \{\mu \circ \Psi^{-1} : \mu \in \Pi\}$. Since Π is a part of $\mathscr{P}(K)$, $\Pi \circ \Psi^{-1}$ is contained in a part of $\mathscr{P}(H)$ (from linearity of the map $\mu \to \mu \circ \Psi^{-1}$). So H has property (1) and since it is metrizable it is, by the first part of the proof, finite dimensional. This contradicts the linear independence of $\{f_n\}$.

LEMMA 4. *Suppose K and H are convex sets and $K^i \neq \varnothing$. Suppose $\Psi: K \to H$ is affine and onto. Then $H^i = \Psi(K^i)$.*

PROOF. Clearly $\Psi(K^i) \subset H^i$. Assume $x \in H^i$. Choose $z' \in K^i$ and let $z = \Psi(z')$. Since $x \in H^i$, $x = \lambda z + (1-\lambda)w$ for some $w \in H$, $0 < \lambda < 1$. Choose $w' \in K$ such that $\Psi(w') = w$. Then if $x' = \lambda z' + (1-\lambda)w'$ we have $\Psi(x') = x$ and $x' \in K^i$ since $z' \in K^i$ and $0 < \lambda < 1$. So $x \in \Psi(K^i)$.

(2)$\Rightarrow$(1). Suppose K is of dimension m and is in fact contained in R^m. If $x \in K^i$ then K contains an open line segment containing x in the direction of each coordinate axis. From the convexity of K we deduce that K and hence K^i contains an open ball in R^m containing x. Hence K^i is open in R^m.

Choose $\{z_i\}$, a countable dense subset of $E(K)$. Let $\mu = \sum_1^\infty (\delta(z_i)/2^i)$ ($\delta(z)$=delta measure at z). We will show $K^i \subset \Phi(\Pi)$ where Π is the part of $\mathscr{P}(K)$ containing μ. Choose $y \in K^i$. Let $\Phi(\mu) = x \in K$. Since $y \in K^i$ we can choose $w \in K^i$ and $1 > \alpha > 0$ so $y = \alpha x + (1-\alpha)w$. Choose $\varepsilon > 0$ so, $\forall\, g \in R^m$,

$$\|g - w\| < \varepsilon \Rightarrow g \in K \quad (\|\cdot\| \text{ is Euclidean norm in } R^m).$$

Choose n so $\{z_1, z_2, \cdots, z_n\}$ is an ε-net for $E(K)$. We claim that $w \in \operatorname{co}\{z_1, z_2, \cdots, z_n\}$. If not $\exists\, \gamma \in R^m$, $\|\gamma\| = 1$ such that $(\gamma, w) > (\gamma, z_i)$ for $1 \leqq i \leqq n$. Now $w + \varepsilon\gamma \in K$. Thus $\exists\, z \in E(K)$ so that

$$(\gamma, z) \geqq (\gamma, w + \varepsilon\gamma) = (\gamma, w) + \varepsilon > (\gamma, z_i) + \varepsilon, \qquad 1 \leqq i \leqq n.$$

It follows that $\|z - z_i\| > \varepsilon$ if $1 \leqq i \leqq n$. This contradicts the choice of n.

So $w \in \operatorname{co}\{z_1, z_2, \cdots, z_n\}$. This provides a measure $\nu \in \mathscr{P}(K)$ such that $\Phi(\nu) = w$ and $\nu \leqq 2^n \mu$. Clearly the probability measure $\alpha\mu + (1-\alpha)\nu$ represents y. It is in Π since $\alpha > 0$ and $\alpha\mu \leqq \alpha\mu + (1-\alpha)\nu \leqq (\alpha + 2^n)\mu$.

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(2) A stronger version of Lemma 1 follows immediately from Bauer's open mapping theorem (to appear in Equationes Mathematicae, see [3, Theorems 5–13]).

(3) There remains the problem for general parts: Find a condition (geometric or topological) on a part Δ of K equivalent to (1).

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