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AN EXTREMAL PROPERTY OF INDEPENDENT RANDOM VARIABLES

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ABSTRACT. In a previous paper the first author proved $Ef(\int_0^t e\,db) \leqq Ef(Mb_t)$, where e is a Brownian functional $\leqq M$ in absolute value and f is a convex function such that the right side is finite. We now prove a discrete analog of this inequality in which the integral is replaced by a martingale transform: $Ef(\sum_1^n d_k y_k) \leqq Ef(M \sum_1^n y_k)$. (The y_j's are independent variables with mean zero, $j \to d_1 y_1 + \cdots + d_j y_j$ is a martingale, and $0 \leqq d_j \leqq M$.) We further show that these inequalities are false if t or n is a stopping time, or if $d_j \not\geq 0$.

1. **Introduction.** Let b_t be Brownian motion, and $e(t, b)$ a non-anticipating function of b_t (see [3] for details). Assume also $|e(t, b)| \leqq M$. In [5] the first author, using a PDE argument, proved the inequality

$$E\left(f\left(\int_0^t e(s, b)\,db_s\right)\right) \leqq E(f(Mb_t)) \tag{1.1}$$

for any convex function $f(x)$ satisfying a certain natural growth condition (see (4.3) below). In this paper, we prove the corresponding result for independent random variables with zero means, which, in contrast to (1.1), requires the analog of $e \geqq 0$. This result is of interest in itself, and can be used to give an alternative proof of (1.1). We also give a third independent proof of (1.1), due to the second author, which consists of only six lines.

Finally, we show that the natural generalizations of (1.1) and its discrete analog—i.e. if $d_k \ngeqq 0$ or t and n are stopping times (see (2.2))—are false. Under these more general conditions, the situation becomes more complicated; see [**1**] for the inequalities that do hold. Inequalities similar to (2.2) (but for the case $f(x) = |x|$) are contained in a paper of Millar [**4**]. See also [2] for another inequality similar to (2.2).

2. **The main result.** Let $y_1, y_2, \cdots, y_n$ be independent random variables with $E(y_k) = 0$, and let $\mathscr{B}_k$ $(0 \leqq k \leqq n)$ be an increasing set of σ-algebras of

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events such that, for each k, (i) $\mathscr{B}(y_1, y_2, \cdots, y_k) \subseteq \mathscr{B}_k$ and (ii) $\{y_{k+1}, \cdots, y_n\}$ is independent of $\mathscr{B}_k$. Finally, let $f(x)$ be a convex function satisfying

$$E\left(\left|f\left(\pm M \sum_1^n |y_k|\right)\right|\right) < \infty$$

and let d_k be random variables such that each d_k is $\mathscr{B}_{k-1}$-measurable. Then

THEOREM 1. *If y_k, d_k and $f(x)$ are as above and*

$$0 \leqq d_k \leqq M \tag{2.1}$$

then

$$E\left(f\left(\sum_1^n d_k y_k\right)\right) \leqq E\left(f\left(M \sum_1^n y_k\right)\right). \tag{2.2}$$

REMARK. The hypotheses of the theorem can be weakened slightly. For example, if the y_k are symmetrically distributed, (2.1) can be replaced by

$$|d_k| \leqq M. \tag{2.3}$$

In general, the theorem holds with (2.3) instead of (2.1) provided

$$E(f(x - \theta y_k)) \leqq E(f(x + \theta y_k)) \tag{2.4}$$

for $1 \leqq k \leqq n$ and all x, $\theta > 0$. Millar [**4**] proved (2.2) for the case $f(x) = |x|$ and symmetric y_k. He also proved (for $f(x) = |x|$) that even in the asymmetric case, with $|d_k| \leqq M$, (2.2) holds if a multiplicative constant 2 is placed on the right side.

EXAMPLES. (1) In particular, if $0 \leqq d_k \leqq M$,

$$E\left(\left(\sum_1^n d_k y_k\right)^{2l}\right) \leqq M^{2l} E\left(\left(\sum_1^n y_k\right)^{2l}\right) \tag{2.5}$$

for integers l. This inequality is of course sharp if $d_k = M$. This inequality is very useful for estimating expressions of this type; the right-hand side can now be expanded as in the proof of the Khinchin inequalities for Rademacher functions. If it had been available, it would have allowed an elementary proof of Theorem 1 in [**6**].

(2) Consider the following "practical" situation. Suppose a gambler, playing a fair roulette wheel, bets d_k on the kth spin of the wheel according to any strategy (randomized or not) which cannot foretell the future. Let T_n be his net winnings up to time n, and let S_n be what he would have won if he had bet a constant amount $M \geqq \max_k d_k$ (e.g. the house limit) each time. Then, for any convex function $f(x)$,

$$E(f(T_n)) \leqq E(f(S_n)).$$

3. **Proofs and counterexamples.** The proof of Theorem 1 depends on the following lemma.

LEMMA. *Let Y be a random variable with mean zero, and $f(x)$ any convex function for which the expectations below exist. Then, for every x,*

$$E(f(x+aY)) \leqq E(f(x+bY)), \qquad 0<a<b. \tag{3.1}$$

PROOF. Without loss of generality assume $x=0$ and $f(0)=0$. Then, if $t=a/b$,

$$f(aY) \leqq (1-t)f(0)+tf(bY)=tf(bY),$$

but $0=f(0)=f(bEY)\leqq Ef(bY)$, so

$$Ef(aY) \leqq tEf(bY) \leqq Ef(bY).$$

(This proof of the Lemma, which is an improvement of our original proof, was suggested by Don Burkholder and the referee. The Lemma is similar to Lemma 1.1 of **[4]**.)

PROOF OF THE THEOREM. By induction (2.2) holds for $1\leqq k\leqq n-1$. Then

$$\begin{aligned} E\left(f\left(\sum_1^n d_k y_k\right)\right) &= E\left(E\left(f\left(\sum_1^{n-1} d_k y_k + d_n y_n\right) \,\middle|\, \mathscr{B}_{n-1}\right)\right) \\ &\leqq E\left(E\left(f\left(\sum_1^{n-1} d_k y_k + M y_n\right) \,\middle|\, \mathscr{B}_{n-1}\right)\right) \\ &= E\left(E\left(f\left(\sum_1^{n-1} d_k y_k + M y_n\right) \,\middle|\, y_n\right)\right) \leqq E\left(f\left(M\sum_1^n y_k\right)\right) \end{aligned}$$

by the Lemma and the induction hypothesis.

COUNTEREXAMPLES. Equation (2.2) cannot hold in general even if $n=1$ and $d_1=-1$. For, if $E(y^2)<\infty$ and $E(f(-y))\leqq E(f(y))$ for all convex $f(x)$ (e.g. $f(x)=\sin\theta x+\theta^2x^2$) then y must be symmetrically distributed. More dramatically, assume $y_1\simeq y_2\simeq 1-z$, where $P(z>t)=e^{-t}$, and

$$f(x)=(e^x-e^2)^+ \qquad (A^+=\max\{A,0\}). \tag{3.2}$$

Then $E(f(y_1-y_2))=\infty$ but $E(f(y_1+y_2))=0$. There is no escape even if $f(x)$ is even, since

$$E((y_1-y_2)^8) \leqq E((y_1+y_2)^8)$$

for $y_1\simeq y_2\simeq y$ holds iff $E(y^3)E(y^5)\geqq 0$, for which it is easy to find counterexamples.

Secondly, assume $P(y_k=\pm 1)=\frac{1}{2}$, $s_k=y_1+y_2+\cdots+y_k$, $s_0=0$, and set $\tau=\min\{n:|s_n|=2\}$, and $d_k=1$ (if $s_{k-1}\leqq 0$) with $d_k=0$ ($s_{k-1}>0$). Then, with probability $1/64$, $s_1,\cdots,s_6=1,0,1,0,1,2$ and $\tau=6$, $\sum_1^\tau d_k y_k=3$. If $f(x)$ is as in (3.2), we obtain a counterexample of (2.2) for τ in place of n. (The choice $f(x)=x^{12}$ also gives a counterexample.)

This example can also be adapted to (1.1). Define stopping times β_n by setting $\beta_0=0$ and, for $n\geqq 0$,

$$\beta_{n+1} = \min\{s+\beta_n: |b(s+\beta_n)-b(\beta_n)|\} = 1.$$

If $e(s,b)=1$ for $\beta_n<s<\beta_{n+1}$ when $b(\beta_n)<0$ and $e(s,b)=0$ for the same range if $b(\beta_n)>0$, then $e(s,b)$ is nonanticipating with respect to $b(t)$. If $\tau=\min\{\beta_n:|b(\beta_n)|=2\}$, then (1.1) is false with τ in place of t, even with $\max_{t\leqq\tau}|b_t|$ in place of b_τ.

We are indebted to Don Burkholder for the idea behind the last two counterexamples.

4. **Other Proofs of (1.1).** By a theorem of McKean [3, p. 29], given any nonanticipating $e(s,b)$ there exists another Brownian motion $C(t)$ such that

$$\int_0^t e(s,b)\,db_s = C(\beta), \qquad \beta = \int_0^t e(s,b)^2\,ds \tag{4.1}$$

and such that β is a stopping time for the Brownian motion $C(t)$. Evidently $\beta\leqq M^2t$; thus $\{0, C(\beta), C(M^2t)\}$ is a martingale by the optional stopping theorem. Hence by Jensen's inequality

$$f(0)\leqq E(f(c(\beta)))\leqq E(f(C(M^2t))) = E(f(Mb_t))$$

since $C(M^2t)\simeq MC(t)$, and hence (1.1) follows.

COROLLARY. *Inequality* (1.1) *also holds even if only*

$$\left(\frac{1}{t}\int_0^t e(s,b)^2\,ds\right)^{1/2}\leqq M.$$

We can also obtain (1.1) from (2.2). For all N and $k\leqq Nt-1$, set

$$d_k = N\int_{(k-1)/N}^{k/N} e(p,b)\,dp, \qquad y_k = b\left(\frac{k+1}{n}\right)-b\left(\frac{k}{n}\right), \tag{4.2}$$

and let $e_N(s,b)$ be step functions defined by $e_N(s,b)=d_k$ for $k\leqq Ns<k+1$; $e_N(s,b)=0$ for $s\geqq [Nt]/N$. The variables $\{d_k, y_k\}$ satisfy the hypotheses of §2 and (2.3), (2.4); hence,

$$E\left(f\left(\int_0^t e_N\,db_s\right)\right)\leqq E(f(Mb([Nt]/N)))\leqq E(f(Mb_t))$$

by the Lemma. On the other hand, $\int_0^t (e_N - e)^2 \, ds \to 0$ a.s. by construction and hence $\int_0^t e_N \, db_s \to \int_0^t e \, db_s$ in probability (see [3]). Finally, since, for all positive θ,

$$E\left(\exp\left(\theta \int_0^t e_N \, db_s - \tfrac{1}{2}\theta^2 \int_0^t e_N^2 \, ds\right)\right) = 1,$$

$P(\int_0^t e_N \, db_s > \lambda) \leqq \exp(-\lambda^2/2M^2t)$, unif. in N. Hence (1.1) follows by the Vitali Convergence Theorem for any convex $f(x)$ satisfying

$$f(x) = O(\exp(|x|^d)), \quad \text{some } d < 2. \tag{4.3}$$

REMARK. This last argument shows that (if $e \geqq 0$) inequality (1.1) holds for any martingale b_t with independent increments with an appropriate modification of (4.3). (If b_t is symmetric e need not be $\geqq 0$.) The special case $f(x) = |x|$, with a multiplicative constant 2 on the right side, appears in [4]. One example would be $b_t = p_t - Ct$, where p_t is a Poisson process with rate C.

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