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AN INTERNAL CHARACTERIZATION OF THE PRIME RADICAL OF A JORDAN ALGEBRA

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ABSTRACT. The prime radical of a Jordan algebra $\mathfrak{A}$ is the set of all very strongly nilpotent elements of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a quadratic Jordan algebra over a commutative, associative ring with 1. (For basic definitions, see [2].) Then an ideal P of $\mathfrak{A}$ is a prime ideal of $\mathfrak{A}$ if, whenever B and C are ideals of $\mathfrak{A}$ and $B\,U_C \subseteq P$, then either $B \subseteq P$ or $C \subseteq P$. The prime radical of $\mathfrak{A}$ is defined to be the intersection of all prime ideals of $\mathfrak{A}$. The concepts of prime ideal and prime radical of a linear Jordan algebra are discussed in [4]. One may apply these concepts to quadratic Jordan algebras without additional difficulty. (See [1].)

The treatment in [4] is incomplete as far as the internal structure of the prime radical of a Jordan algebra is concerned. One can show, among other things, that there exists a unique prime radical $P(\mathfrak{A})$ of a Jordan algebra $\mathfrak{A}$, which is the intersection of all prime ideals of $\mathfrak{A}$, and the prime radical of the quotient algebra $\mathfrak{A}/P(\mathfrak{A})$ is 0. Moreover, if the radical of $\mathfrak{A}$ is zero then $\mathfrak{A}$ can be represented as a subdirect sum of prime algebras. However, very little was said about the prime radical itself besides that every element of $P(\mathfrak{A})$ is nilpotent, i.e., $P(\mathfrak{A})$ is a nil ideal of $\mathfrak{A}$.

For associative rings J. Lambek gives an internal characterization of the prime radical $P(R)$ of a ring R. He shows, in [3], that an element x of R is in $P(R)$ if and only if x is strongly nilpotent. In this paper we shall show a similar characterization for the prime radical of Jordan algebras.

Let $\mathfrak{A}$ be a Jordan algebra over Φ. If $a \in \mathfrak{A}$, then $[a]$ denotes the principal ideal of $\mathfrak{A}$ generated by a. A sequence $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$ of elements of $\mathfrak{A}$ will be called an m-sequence if $x_{n+1} \in J\,U_{x_n}$ and will be called an M-sequence if $x_{n+1} \in [x_n]\,U_{[x_n]}$. An element x is called a strongly nilpotent element if every m-sequence beginning with $x_0 = x$ is ultimately zero and is called very strongly nilpotent if every M-sequence beginning with $x_0 = x$ is ultimately zero.

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LEMMA 1. *Every m-sequence in $\mathfrak{A}$ contains an M-subsequence.*

PROOF. Let $S=\{x_0, x_1, x_2, \cdots\}$ be an m-sequence of $\mathfrak{A}$. Then the subsequence $S'=\{x'_0, x'_1, x'_2, \cdots, x'_n, \cdots\}$ where $x'_n=x_{2n}$ is a sequence of $\mathfrak{A}$. Since S is an m-sequence for each u, there exists an element $y_n \in \mathfrak{A}$ such that $x_{n+1}=y_n U_{x_n}$. Thus for each n, we see $x'_{n+1}=x_{2n+2}=y_{2n+1} U_{x_{2n+1}}=(y_{2n+1} U_{x_{2n}} U_{y_{2n}}) U_{x_{2n}}$ is an element of $[x'_n] U_{[x'_n]}$. Hence S' is an M-sequence. We see easily:

COROLLARY. *Every very strongly nilpotent element of $\mathfrak{A}$ is a strongly nilpotent element of $\mathfrak{A}$. Every strongly nilpotent element of $\mathfrak{A}$ is a nilpotent element.*

THEOREM 1. *The prime radical $P(\mathfrak{A})$ of $\mathfrak{A}$ is the set of all very strongly nilpotent elements in $\mathfrak{A}$.*

PROOF. Let a be an element in $\mathfrak{A}$ but not in $P(\mathfrak{A})$. Then there exists a prime ideal P of $\mathfrak{A}$ which does not contain a. The set complement P^c of P is a Q-system (see [4]) which contains a. Let $a_0=a$ and a_1 be a nonzero element in $[a_0] U_{[a_0]} \cap P^c$ and, inductively, for each integer n there exists a nonzero element a_{n+1} in $[a_n] U_{[a_n]} \cap P^c$. Moreover, no member of the infinite sequence $S=\{a_0, a_1, a_2, \cdots\}$ can be zero so that a is not a very strongly nilpotent element.

If a is not a very strongly nilpotent element, then there exists an infinite sequence $T=\{a_0, a_1, a_2, \cdots\}$, $a_0 = a$, $a_{n+1} \in [a_n] U_{[a_n]}$, $a_k \neq 0$. Let P be a maximal ideal with respect to the property that is disjoint from T (use Zorn's lemma). If one shows that P is a prime ideal of $\mathfrak{A}$ which misses a, then a is not contained in the prime radical $P(\mathfrak{A})$.

Let B and D be ideals of $\mathfrak{A}$ not contained in P; then by the maximality of P and the fact that both the ideals $B+P$ and $D+P$ properly contain P, there exist a_i in $B+P$ and a_j in $D+P$. For convenience, we let $i \leqq j$, then $a_j \in [a_i]$ and $[a_j] \subseteq [a_i]$ so that $a_{j+1} \in [a_j] U_{[a_j]} \subseteq [a_j] U_{[a_i]} \subseteq (D+P) U_{B+P} \subseteq D U_B + P$ and $a_{j+1} \notin P$. Hence $D U_B \nsubseteq P$, and P is a prime ideal of $\mathfrak{A}$.

COROLLARY. *If K is an ideal of $\mathfrak{A}$, then the prime radical of K consists of the set of all elements of $\mathfrak{A}$ which are very strongly nilpotent mod K.*

If R is an associative algebra over Φ and x is in R, then x is strongly nilpotent if every sequence $\{x_0, x_1, x_2, \cdots\}$, such that $x_0=x$ and $x_{n+1}=x_n r_n x_n$ for some r_n in R, is ultimately zero. J. Lambek shows that the prime radical of R is the set of strongly nilpotent elements of R.

It is well known that one may construct a quadratic Jordan algebra R^+ on R by defining the operator $U_a: y \rightarrow aya$, cf. [2]. If a is an element in R (and so is in R^+), we shall say a is strongly R-nilpotent if it is a strongly nilpotent element in R. Similarly we shall say a is strongly (very strongly) R^+-nilpotent if it is a strongly (very strongly) nilpotent element in R^+.

THEOREM 2. *Let R be an associative algebra over Φ and a be an element in R. Then the following statements are equivalent*:

(1) *a is strongly R-nilpotent*;

(2) *a is strongly R^+-nilpotent*;

(3) *a is very strongly R^+-nilpotent.*

PROOF. We shall only show (1) implies (3), for the equivalence of (1) and (2) follows immediately from the definition. (3) implies (2) was shown by Lemma 1 and its corollary.

Since the prime radical $P(R)$ of R is the set of all strongly R-nilpotent elements as pointed out by Lambek and the prime radical $P(R^+)$ of R^+ is the set of all very strongly R^+-nilpotent elements in Theorem 1, we need only to show $P(R)\subseteq P(R^+)$.

Let $\mathfrak{S}=\{K|K$ is an ideal of R contained in $P(R^+)\}$. Then, by Zorn's lemma, there exists a maximal element B in this collection. We note that B is an ideal of R and if K is an ideal of R such that $K^2\subseteq B$, then if one takes K as an ideal of R^+, $K\,U_K \subseteq K^2\subseteq B\subseteq P(R^+)$. So $K\subseteq P(R^+)$. Thus, by the maximality of B, $K\subseteq B$. Therefore B is a semiprime ideal of R and $P(R)\subseteq B\subseteq P(R^+)$.

As a consequence of this argument we have a simple proof for a result of Erickson and Montgomery, [**1**, Theorem 4].

THEOREM 3 (ERICKSON AND MONTGOMERY). *Let R be an associative algebra over Φ. Then the prime radical $P(R)$ of R coincides with the prime radical $P(R^+)$ of the Jordan algebra R^+.*

PROOF. Both $P(R)$ and $P(R^+)$ consist of exactly the set of all strongly nilpotent elements of R (or R^+).

We shall now consider an associative algebra R with an involution $x\to x^*$. Consider that the space S of $*$-symmetric elements is a quadratic Jordan algebra. We have

THEOREM 4. *Let R be an associative algebra with involution and a be an element in S, the space of symmetric elements. Then the following statements are equivalent*:

(1) *a is a strongly nilpotent element in S*;

(2) *a is a very strongly nilpotent element in S*;

(3) *a is very strongly R^+-nilpotent*;

(4) *a is strongly R-nilpotent.*

We shall prove the following lemma first.

LEMMA 2. *If $a\in S$ and $aSa\subseteq P(R)$, then $a\in P(R)$.*

PROOF. For any $x, y\in R$, $a(x^*ax)a\in aSa\subseteq P(R)$ and $a(xay+y^*ax^*)a\in aSa\subseteq P(R)$. Thus $(axa)y(axa)=a(xay+y^*ax^*)axa-ay^*(ax^*axa)\in P(R)$.

Hence $(axa)R(axa) \subseteq P(R)$ which yields $axa \in P(R)$ for $P(R)$ is a semiprime ideal in R. Now $axa \in P(R)$ for all $x \in R$ yields that $aRa \subseteq P(R)$ and so $a \in P(R)$ by the same reason.

Proof of the theorem. We shall show (1) implies (4). If a is not strongly R-nilpotent, then, by [**1**], $a \notin P(R)$. Let $a_0 = a$, then, by Lemma 2, $a_0 S a_0 \nsubseteq P(R)$. Thus there exists $s_0 \in S$ such that $a_1 = a_0 s_0 a_0 \notin P(R)$. If $a_n \notin P(R)$ is obtained then there exists an element s_n in S so that $a_{n+1} = a_n s_n a_n \notin P(R)$ by the same argument. Hence one finds an m-sequence $\{a_0, a_1, a_2, \cdots\}$ in S which begins with a and never terminates at zero. Therefore a is not a strongly nilpotent element in the Jordan algebra S.

The proof of the theorem is essentially completed, for the implications (4) implies (3); (3) implies (2) and (2) implies (1) follow from the definition and Lemma 1 easily.

The following theorem of Erickson and Montgomery [**1**, Theorem 3] can now be proven in a very simple manner.

Theorem 5 (Erickson and Montgomery). *Let R be an associative algebra with involution $*$ and S be the set of $*$-symmetric elements of R. Then the prime radical $P(S)$ of the Jordan algebra S is the intersection $P(S) = S \cap P(R)$ of S with the prime radical of $P(R)$ of the associative algebra R.*

Proof. Both consist of the set of strongly nilpotent elements of S.

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