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COMPACT INDEPENDENT SETS AND HAAR MEASURE

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ABSTRACT. This is proved: Let H be a closed nondiscrete subgroup of an LCA group G, $x \in G$, and $E \subseteq G$ a σ-compact independent subset of G; then $H \cap (x+G_pE)$ has zero H-Haar measure. This generalizes a result in Rudin, *Fourier analysis on groups*; the proof here is quite different from that given by Rudin.

THEOREM 1. *Suppose that H is a nondiscrete LCA group which is a subgroup of a group G, $x \in G$ and that $E \subseteq G$ is an independent set. If*

(i) *G is a LCA group such that H with its topology is a closed subgroup in G, and*

(ii) *E is σ-compact;*

then

$$H \cap (x + G_pE) = H \cap \left(x + \bigcup_{n=1}^{\infty} n(E \cup -E)\right)$$

has zero Haar measure in H.

Theorem 1 generalizes the case $H=G$, $x=0$, E compact, given [**1**, 5.3.6] by Rudin.

The hypotheses (i) and (ii) of Theorem 1 may be modified in several ways; here is one of them.

THEOREM 2. *Suppose that H is a nondiscrete LCA group, which is a subgroup of a group G, $x \in G$, and that $E \subseteq G$ is an independent set. If*

(iii) *H is σ-compact and $H \cap (x+n(E \cup -E))$ is a Borel subset of H for each $n=1, 2, \cdots$,*

then $H \cap (x+G_pE)$ has zero Haar measure in H.

Comments. Theorem 1 becomes false (in general) if (a) "closed" is omitted from (i) or if (b) "σ-compact" is omitted from (ii). Theorem 2 becomes false (in general) if (c) "σ-compact" is omitted from (iii), or if (d) "Borel" is omitted from (iii).

In cases (a)–(c) we consider $H_a = \boldsymbol{R}_d \times \boldsymbol{R}$, and $E_1 \subseteq \boldsymbol{R}$ a compact perfect independent set. Then the natural embedding of H_a in $\boldsymbol{R} \times \boldsymbol{R}$ and $E = E_1 \times \{0\}$ yield (a): $H = H_a$; $G = \boldsymbol{R} \times \boldsymbol{R}$; E has infinite Haar measure in H.

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For (b) and (c) we choose $H=H_a=G$, and $E=E_1\times\{0\}$. (This example was suggested to us by the referee.) For case (d), let $E\subseteq \boldsymbol{R}$ be a maximal independent set. It is easy to see that $n(E\cup -E)$ will not be measurable (much less Borel) for some n, and $\boldsymbol{R}=\bigcup_{n=1}^{\infty}(1/n)G_pE$.

This paper is divided: In §1 notation is given; in §2 some easy basic lemmas are proved; in §3, Theorem 1 is proved; the proof of Theorem 2 follows from Lemma 2, using the arguments of §3, and the observation that (iii) is exactly what is needed to obtain Lemma 2.

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1. **Notation and definitions.** A subset E of a group G is *independent* if whenever $x_1, \cdots, x_m \in E$ are distinct, $n_1, \cdots, n_m \in Z$, and $\sum n_j x_j=0$ then $n_1x_1=\cdots=n_mx_m=0$. $G_pE=\bigcup_1^\infty n(Q)$, where $Q=E\cup -E$, $1(Q)=Q$ and $n(Q)=(Q)+(n-1)(Q)$, $n=2, 3, \cdots$.

We follow the conventions of [**1**].

The identity of a group K is denoted by 0_K. h will denote Haar measure on the LCA group H. By $X\setminus Y$ we mean $\{x\in X: x\notin Y\}$.

2. Some easy lemmas.

LEMMA 1. *Let G be an LCA group, and let $U\subseteq G$ be open. Then $G_pU=G_p\bar{U}$.*

PROOF. $G_pU\subseteq G_p\bar{U}$ and $\bar{U}\subseteq U+U-U\subseteq 3(U\cup -U)\subseteq G_pU$. Thus $G_p\bar{U}\subseteq G_pU$. Q.E.D.

COROLLARY. *If U is open and is σ-compact, then G_pU is an open σ-compact subgroup.*

PROOF. Obvious.

LEMMA 2. *Let H be an LCA group which is (qua group) a subgroup of a group G. Let $E\subseteq G$ be independent, and $x\in G$. Suppose H is σ-compact and $H\cap(x+nQ)$ is a Borel set in H, for each integer $n\geqq 1$, and*

$$h(H\cap(x+G_pE))\neq 0.$$

Then there exists an integer $k\geqq 2$ such that $kQ\cap H$ is a neighborhood (in H) of 0_H, where $Q=E\cup -E$.

PROOF. $(x+G_pE)\cap H=\bigcup_{n=1}^\infty (x+nQ)\cap H$, so for some $n\geqq 1$,

$$h((x+nQ)\cap H)\neq 0.$$

Since h is a σ-finite regular measure (H is σ-compact), there is a compact subset $X\subseteq(x+nQ)\cap H$ such that $h(X)\neq 0$. Using the argument of

[**1**, 1.4.2a], we let f be the characteristic function of X. Then $f\in L^2(H)\cap L^1(H)$ and $f*\tilde{f}(0)=\|f\|_2^2=h(X)\neq 0$, while $f*\tilde{f}$ is continuous on H. The support of $f*\tilde{f}$ is contained in $X-X\subseteq((x+nQ)-(x+nQ))\cap H\subseteq(2nQ)\cap H$. Thus $(2nQ)\cap H$ is a neighborhood (in H) of 0_H. Q.E.D.

LEMMA 3. *If H is a compact abelian group, and U is a symmetric neighborhood of the identity 0_H, then there is an integer n such that $H^1=nU$ is a compact open subgroup of finite index in H.*

PROOF. $\bigcup_{j=1}^{\infty} jU=H^1$ is an open and closed subgroup, and hence compact, since H is compact. Therefore, for some n, $\bigcup_{j=1}^{n} jU=H^1=nU$. Since H is compact and H/H^1 is both compact and discrete, H^1 has finite index. Q.E.D.

LEMMA 4. *Let E be a compact infinite independent set, and $P\subseteq E$ a finite subset. Let $Q=E\cup -E$, $Q'=Q\backslash(P\cup -P)$. Then mQ' has interior in mQ for every integer $m=1,2,\cdots$.*

PROOF. It is easy to see (using the independence of E) that

$$mQ=(mQ')\cup((m-1)Q'+(P\cup -P))\cup\cdots \cup(Q'+(m-1)(P\cup -P))\cup(m(P\cup -P)).$$

Thus

$$mQ'\supseteq mQ\backslash((m-1)Q+m(P\cup -P)). \tag{1}$$

Since $mQ\cap[((m-1)Q)+(m(P\cup -P))]$ is closed, the set on the right-hand side of (1) is open. That set is not empty, since if $x_1,\cdots,x_m\in E\backslash P$ are distinct, then

$$x_1+\cdots+x_m\notin(m-1)Q+m(P\cup -P)$$

(otherwise E could not be independent). Q.E.D.

3. **Proof of Theorem 1.** *Reduction to the case H is σ-compact.* Let U be an open subset of G, with σ-compact closure, which contains $E\cup\{x\}$. Then $K=G_pU$ is (by Lemma 1 and its corollary) a σ-compact open subgroup of G, and

$$H\cap(x+G_pE)=(H\cap K)\cap(x+G_pE).$$

Because H is closed in G, $H\cap K$ is σ-compact. The Haar measure of $H\cap K$ is the restriction of that of H to $H\cap k$ so we may assume that H is σ-compact.

Reduction to the case: E is compact. Since E is σ-compact, we may write $E=\bigcup_{j=1}^{\infty} E_j$, where each E_j is compact. Then

$$k(E\cup -E)=\bigcup_{n=1}^{\infty} k\left(\bigcup_{j=1}^{n}(E_j\cup -E_j)\right).$$

Then

$$x + G_p E = \bigcup_{n=1}^{\infty} \left(x + G_p \bigcup_{j=1}^{n} E_j \right).$$

We see that $h(x+G_pE)\neq 0$ implies that $h(x+G_p \bigcup_{j=1}^{n} E_j)\neq 0$ for some n. Thus, we may assume that E is compact.

Using Lemma 2, we see that we may now assume that (for some $k\geqq 2$) $x=0_H$ and $U=kQ\cap H$ is a neighborhood in H of 0_H. [Lemma 2 may be applied because $H\cap(x+nQ)$ is a compact subset of H.]

We now have two cases.

Case I: H is compact. Lemma 3 tells us that H^1 (as defined, using $U=kQ\cap H$, in Lemma 3) is a compact open subgroup of H, and that there is an integer $n\geqq 1$ such that $nU=H^1$. Hence

$$H^1 = (nkQ) \cap H = (nkQ) \cap H^1.$$

Now, H^1 contains an infinite number of distinct elements $x_1, x_2, \cdots$, because H^1 is nondiscrete.

Since $x_j \in nkQ$, there exist integers M_j and α_{jm}, and elements $p_{jm} \in E$ such that

$$\alpha_{jm}p_{jm} \neq 0_G, \qquad p_{jm} \neq p_{jm'}, \quad m \neq m', \quad j = 1, 2, \cdots \tag{2}$$

$$\sum_{m=1}^{M_j} |\alpha_{jm}| \leqq nk \quad \text{and} \quad x_j = \sum_{m=1}^{M_j} \alpha_{jm}p_{jm}. \tag{3}$$

If $P=\{p_{jm}: 1\leqq m\leqq M_j, j=1, 2, \cdots\}$ is a finite subset of E then, by (3), there can be only a finite number of distinct x_j. Hence P is infinite. By applying induction to (2) and (3) we can find integers $j(1), j(2), \cdots$, $m(1), m(2), \cdots$ such that

$$p_{j(s),m(s)} \notin \{p_{j(t),m}: 1 \leqq m \leqq M_{j(t)},\ t \neq s,\ t = 1, 2, \cdots\}. \tag{4}$$

[Here are the details of the induction: For each $j=1, 2, \cdots$, there can be at most nk distinct elements of P which belong to

$$\{p_{jm}: 1 \leqq m \leqq M_j \leqq nk\} = P_j,$$

that is which appear in the expansion (3) of x_j; also $nk\,P_j$ can contain only a finite number (a crude estimate is (card $P_j)^{2nk}\leqq(nk)^{2nk}$) of elements of H^1. We choose $j(1)=1$, and apply the preceding sentence to produce $1\leqq m(1)\leqq M_1$ and an infinite subset $\{x_j^{(1)}\}$ of $\{x_j\}$ such that the expansion (3) for each $x_{k(1,j)}=x_j^{(1)}$ does not contain $p_{1,m(1)}$.

We now repeat the above process with $\{x_j^{(1)}\}$, obtaining $j(2)=k(1, 1)$, $1\leqq m(2)\leqq M_{j(2)}$, and $\{x_j^{(2)}\}\leqq\{x_j^{(1)}\}$ and an infinite subset $\{x_j^{(2)}\}\subseteq\{x_j^{(1)}\}$ such that the expansion (3) for each $x_{k(2,j)}=x_j^{(2)}$ does not contain $p_{j(2),m(2)}$.

Continuing in this way, we see that (4) may be satisfied.]

From the independence of E and (2)–(4) we see that $x_{j(1)}+\cdots+x_{j(kn+1)}\notin knQ$, so $(kn+1)H^1\nsubseteq knQ$, which contradicts our choice of kn. This means that $h(x+G_pE)=0$, if H is compact.

Case II: H not compact. We shall reduce to the preceding case, by using the well-known [**1**, 2.4.2] result that a locally compact abelian group A generated by a compact neighborhood U of its identity contains a closed discrete subgroup B of the form Z^p such that A/B is compact.

Choose a compact symmetric neighborhood $V\subseteq H$ of 0_H such that $V\subseteq kQ\cap H$, and let $A=\bigcup_{n=1}^{\infty} nV$. Then A is a σ-compact open subgroup of H and $A\cap kQ\supseteq V$. Thus A satisfies the hypotheses of Theorem 1, and is generated by a compact neighborhood of its identity. Of course, Haar measure on A is precisely that of H restricted to A.

Let B be a subgroup with the properties above. Since B is finitely generated and $A\subseteq G_pE$, there exists a finite set $P\subseteq E$ such that $B\subseteq G_pP$. Set $E'=E\backslash P$, $Q'=E'\cup -E'$ and $H^2=A\cap G_pE'$. We claim H^2 is a compact open subgroup of A. If this is established, then the Haar measure of H^2 will be that of A restricted to H^2, and we may apply Case I to H^2.

H^2 *is open*. Since $A\subseteq G_pE=\bigcup_{m=1}^{\infty} mQ$ and each mQ' has interior in mQ (by Lemma 4) for $m=1,2,\cdots$, $A\cap mQ'$ has interior in $A\cap mQ$ for $m=1,2,\cdots$. Since $A\cap kQ$ is open in A, $A\cap kQ'$ has interior in A. Because $A\cap kQ'$ is symmetric, $A\cap 2kQ'=2(A\cap kQ')$ contains a neighborhood of 0_A in A. Therefore $A\cap G_pE'=H^2$ is open in A.

H^2 *is compact*. Consider the sequence of maps $A\to A/B\to A/C$ where $C=A\cap G_pP$. Now $C\cap H^2=\{0_H\}$, since E is independent, and $C+H^2=A$. Because H^2 is open, C must be closed. Now $A/C=(C+H^2)/C\simeq H^2$, as groups. On the other hand $C\supseteq B$, and C is closed, so $A/B\to A/C=H^2$ is continuous, and A/B is compact. Hence H^2 is compact.

[*Note.* That C is closed is proved as follows: $C+H^2=A$, so if $x\in\bar{C}\backslash C$, then x has a neighborhood of the form $c+H^2$, where $c\in C$. Now, there must be an element $d\neq c$, $d\in C$ in $c+H^2$, since $x\notin C$. But $(d+H^2)\cap(c+H^2)=\varnothing$ since $C\cap H^2=\{0\}$. Hence $\bar{C}\backslash C=\varnothing$.]

This completes the reduction to Case I, and the proof of Theorem 1 is complete.

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