

EXISTENCE THEOREMS FOR SUM AND PRODUCT INTEGRALS

JON C. HELTON

ABSTRACT. Necessary and sufficient conditions on a function G are determined for the integrals

$$\int_a^b HG, \quad \prod_a^b (1 + HG), \quad \int_a^b \left| HG - \int HG \right| = 0$$

and

$$\int_a^b \left| 1 + HG - \prod (1 + HG) \right| = 0$$

to exist, where H and G are functions from $R \times R$ to R and H is restricted by one or more of the limits $H(p^-, p)$, $H(p^-, p^-)$, $H(p, p^+)$ and $H(p^+, p^+)$. Furthermore, the conditions on G are sufficient for the existence of these integrals when H and G have their range in a normed complete ring N .

All integrals and definitions are of the subdivision-refinement type, and functions are from $R \times R$ to R or to N , where R represents the set of real numbers and N represents a ring which has a multiplicative identity element denoted by 1 and has a norm $|\cdot|$ with respect to which N is complete and $|1|=1$. Unless noted otherwise, functions are from $R \times R$ to R . If $\{x_q\}_0^n$ is a subdivision of $[a, b]$, then the statement that J is a modified refinement of $\{x_q\}_0^n$ means there exist sequences $\{y_q\}_1^n$, $\{z_q\}_1^n$ and $\{L_q\}_1^n$ such that $x_{q-1} < y_q < z_q < x_q$, L_q is a subdivision of $[y_q, z_q]$ and $J = \bigcup_1^n L_q$. Furthermore, if $D = \{x_q\}_0^n$ is a subdivision of $[a, b]$, then $D(I) = \{[x_{q-1}, x_q]\}_1^n$, and if J is a modified refinement of D , then $J(I) = \bigcup_1^n L_q(I)$. The statements that G is bounded and $G \in OB^\circ$ on $[a, b]$ mean there exists a subdivision D of $[a, b]$ and a number B such that if J is a refinement of D then

(1) $|G(u)| < B$ for $u \in J(I)$, and

(2) $\sum_{J(I)} |G| < B$,

respectively. Further, $G \in AZ$ on $[a, b]$ only if G is bounded on $[a, b]$ and if $\varepsilon > 0$ then there exists a subdivision D of $[a, b]$ such that if J is a modified refinement of D then $\sum_{J(I)} |G| < \varepsilon$. The function $G \in OA^\circ$ or OM° on

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$[a, b]$ if and only if $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$ or $\prod_a^b (1 + G)$ exists and $\int_a^b |1 + G - \prod (1 + G)| = 0$, respectively.

The symbols $H(p^-, p)$, $H(p, p^+)$, $H(p^-, p^-)$ and $H(p^+, p^+)$ represent $\lim_{x \rightarrow p^-} H(x, p)$, $\lim_{x \rightarrow p^+} H(p, x)$, $\lim_{x, y \rightarrow p^-} H(x, y)$ and $\lim_{x, y \rightarrow p^+} H(x, y)$, respectively. If H is a bounded function on $[a, b]$, then

(1) $H \in OL^1$ on $[a, b]$ only if $H(p^-, p^-) = H(p^+, p^+) = 0$ for each p in $[a, b]$,

(2) $H \in OL^2$ on $[a, b]$ only if $H(p^-, p^-)$ and $H(p^+, p^+)$ exist for each p in $[a, b]$,

(3) $H \in OL^3$ on $[a, b]$ only if $H(p^-, p) = H(p, p^+) = 0$ for each p in $[a, b]$,

(4) $H \in OL^4$ on $[a, b]$ only if $H(p^-, p)$ and $H(p, p^+)$ exist for each p in $[a, b]$, and

(5) $H \in OL^{ij}$ on $[a, b]$ only if $H \in OL^i \cap OL^j$ on $[a, b]$.

Note that OL^{24} is the same as the set OL° studied by B. W. Helton [3, p. 493].

In the following we show that the conditions in Theorem 2 are necessary and sufficient conditions for functions from $R \times R$ to R ; furthermore, with the assistance of Theorem 1, a number of theorems, each of which gives three equivalent conditions for a product HG of functions to belong to OA° or OM° , are obtained. Whenever functions from $R \times R$ to N are considered, in each theorem statement (3) implies each of statements (1) and (2).

THEOREM 1. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then $G \in OA^\circ$ on $[a, b]$ if and only if $G \in OM^\circ$ on $[a, b]$ [2, Theorem 3.4, p. 301].*

THEOREM 2. *If H and G are functions from $R \times R$ to N such that $H \in OL^\circ$, $G \in OA^\circ$ and $G \in OB^\circ$ on $[a, b]$, then GH and $HG \in OA^\circ$ and OM° on $[a, b]$ [3, Theorem 2, p. 494].*

LEMMA 3.1. *If $G \notin OB^\circ$ on $[a, b]$, then there exists $H \in OL^{13}$ such that HG is nonnegative and $HG \notin OB^\circ$ on $[a, b]$.*

PROOF. Observe that the desired function can be constructed if there exists $p \in [a, b]$ such that either $G(x, p)$ as $x \rightarrow p^-$ or $G(p, x)$ as $x \rightarrow p^+$ is unbounded. Therefore, assume these bounds exist. There exists a sequence $\{D_n\}_{n=1}^\infty$ of subdivisions of $[a, b]$ such that

- (1) D_{n+1} is a refinement of D_n ,
- (2) if $u \in D_n(I)$, then $u \notin D_{n+1}(I)$, and
- (3) $\sum_{D_n(I)} |G| > n^2$.

Let H be the function such that

- (1) if $u \in D_n(I)$, then $H(u) = 1/n$ if $G(u) \geq 0$ and $H(u) = -1/n$ if $G(u) < 0$, and
- (2) if $u \notin \bigcup_{n=1}^\infty D_n(I)$, then $H(u) = 0$.

Thus, $H \in OL^{13}$ and HG is nonnegative. Furthermore, since $G(x, p)$ as $x \rightarrow p^-$ and $G(p, x)$ as $x \rightarrow p^+$ are bounded for each $p \in [a, b]$, it follows that $HG \notin OB^\circ$.

LEMMA 3.2. *If $G \notin OB^\circ$ on $[a, b]$, then there exists $H \in OL^{13}$ such that $HG \notin OA^\circ$ and $HG \notin OM^\circ$ on $[a, b]$.*

PROOF. It follows from Lemma 3.1 that there exists a function $H \in OL^{13}$ such that HG is nonnegative and $HG \notin OB^\circ$. Hence, $HG \notin OA^\circ$ and $HG \notin OM^\circ$.

THEOREM 3. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^1$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^1$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in OL^3$ and $G \in OB^\circ$ on $[a, b]$.*

PROOF. Since it is possible for $H \in OL^1$ without $H \in OL^4$, it is necessary that $G \in OL^3$ if (3) is to imply either (1) or (2). Furthermore, it follows from Lemma 3.2 that $G \in OB^\circ$ is a necessary condition for (3) to imply either (1) or (2). In order to show that (3) implies (1) and (2), I will show that if $H \in OL^1$, then $\int_a^b HG = 0$, and hence, $HG \in OA^\circ$ and, from Theorem 1, $HG \in OM^\circ$. Let $H \in OL^1$ and suppose $\varepsilon > 0$. There exists a subdivision $E = \{x_q\}_0^n$ of $[a, b]$ and a number B such that

(1) if J is a refinement of E , then $\sum_{J(I)} |G| < B$ and if $u \in J(I)$ then $|H(u)| < B$, and

(2) if J is a modified refinement of E and $u \in J(I)$, then $|H(u)| < \varepsilon/2B$.

Furthermore, there exists a positive number δ such that if u is a subinterval of either $[p - \delta, p]$ or $[p, p + \delta]$ for some element p of E then $|G(u)| < \varepsilon/4nB$. Let D be the subdivision of $[a, b]$ such that

$$D = E \cup \{x_q + \delta\}_0^{n-1} \cup \{x_q - \delta\}_1^n,$$

and suppose J is a refinement of D . Let $K(I)$ be the subset of $J(I)$ such that $u \in K(I)$ only if u has an element of E as an endpoint. Thus,

$$\sum_{J(I)} |HG| = \sum_{K(I)} |HG| + \sum_{J(I)-K(I)} |HG| < (2nB)(\varepsilon/4nB) + (\varepsilon/2B)B = \varepsilon.$$

Note that condition (3) of Theorem 3 is not a sufficient condition for $G \in OA^\circ$ on $[a, b]$. For example, consider the function G such that $G(x, y) = y - x$ if x is rational and $G(x, y) = 2(y - x)$ if x is irrational.

THEOREM 4. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^2$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^2$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in OL^3$, $G \in OB^\circ$ and $G \in OA^\circ$ on $[a, b]$.*

PROOF. It follows as in Theorem 3 that each of (1) and (2) implies that $G \in OL^3$ and $G \in OB^\circ$. Further, since $H \equiv 1$ is in OL^2 , it follows that $G \in OA^\circ$ if (1) is true and $G \in OM^\circ$ if (2) is true. Thus, it follows by using Theorem 1 that $G \in OA^\circ$ if (2) is true. Hence, we only need to show that (3) implies (1) and (2). We now show that if $H \in OL^2$ then $HG \in OA^\circ$; then it follows from Theorem 1 that $G \in OM^\circ$. Suppose $H \in OL^2$ and $\varepsilon > 0$. There exists a subdivision $E = \{x_q\}_0^n$ of $[a, b]$, a number B , and a set $\{a_q\}_1^n$ such that

(1) if J is a refinement of E then $\sum_{J(I)} |G| < B$, and if $u \in J(I)$ then $|H(u)| < B$,

(2) if $x_{q-1} < x < y < x_q$ then $|a_q - H(x, y)| < \varepsilon/6B$,

(3) if L_q is a subdivision of $[x_{q-1}, x_q]$ then $\sum_1^n |G(x_{q-1}, x_q) - \sum_{L_q(I)} G| < \varepsilon/6B$, and

(4) $|a_q| < B$.

There exists a positive number δ such that if r and s are in E then

(1) $[r - \delta, r + \delta]$ and $[s - \delta, s + \delta]$ do not intersect, and

(2) if J is a subdivision of $[r - \delta, r + \delta]$ then

$$\sum_{J(I)} |G| < \varepsilon/6(n+1)B,$$

where $r - \delta = a$ if $r = a$ and $r + \delta = b$ if $r = b$.

We now divide the proof into two parts. In the first part we use the Cauchy criterion to show that $\int_a^b HG$ exists, and in the second part we show that $\int_a^b |HG - \int HG| = 0$.

PART I. Let D be the subdivision of $[a, b]$ such that

$$D = E \cup \{x_q - \delta\}_1^n \cup \{x_q + \delta\}_0^{n-1},$$

and suppose J is a refinement of D . Further, let $K(q)$ and $L(q)$ be the subsets of $D(I)$ and $J(I)$, respectively, such that $u \in K(q)$ or $L(q)$ only if it is a subinterval of $[x_q - \delta, x_q + \delta]$, and let $M(q)$ and $N(q)$ be the subsets of $D(I)$ and $J(I)$, respectively, such that $u \in M(q)$ or $N(q)$ only if it is a subinterval of $[x_{q-1} + \delta, x_q - \delta]$. Thus,

$$\begin{aligned} \left| \sum_{D(I)} HG - \sum_{J(I)} HG \right| &= \left| \sum_0^n \left[\sum_{K(q)} HG - \sum_{L(q)} HG \right] \right. \\ &\quad + \sum_1^n \left[\sum_{M(q)} HG - \sum_{N(q)} HG \right] \Big| \leq \sum_0^n \left[\sum_{K(q)} |HG| + \sum_{L(q)} |HG| \right] \\ &\quad + \sum_1^n \left[\sum_{M(q)} |H - a_q| |G| + \sum_{N(q)} |H - a_q| |G| \right] \\ &\quad + \sum_1^n |a_q| \left| \sum_{M(q)} G - \sum_{N(q)} G \right| \\ &< B[n+1][\varepsilon/6(n+1)B + \varepsilon/6(n+1)B] + [\varepsilon/6B][2B] + B[\varepsilon/6B] < \varepsilon. \end{aligned}$$

PART 2. Let $D, J, L(q)$ and $N(q)$ be defined as in Part 1. Thus,

$$\begin{aligned}
 & \sum_{J(I)} \left| HG - \int HG \right| \\
 &= \sum_0^n \sum_{L(q)} \left| HG - \int HG \right| + \sum_0^n \sum_{N(q)} \left| HG - \int HG \right| \\
 &\leq \sum_0^n \sum_{L(q)} \left[|HG| + \left| \int HG \right| \right] \\
 &\quad + \sum_1^n \left[\sum_{N(q)} |H - a_q| |G| + \sum_{N(q)} \left| \int (H - a_q) G \right| \right] \\
 &\quad + \sum_1^n \left[\sum_{N(q)} |a_q| \left| G - \int G \right| \right] \\
 &< B[n+1][\varepsilon/6(n+1)B + \varepsilon/6(n+1)B] + [\varepsilon/6B][2B] + B[\varepsilon/6B] < \varepsilon.
 \end{aligned}$$

If we restrict our consideration to real-valued functions, then the existence of $\int_a^b F$ is a sufficient condition for $F \in OA^\circ$ [1, Theorem 1, p. 155]. More generally, the existence of $\int_a^b F$ is sufficient for $F \in OA^\circ$ provided the range of F is restricted to certain rings [2, Theorem 4.1, p. 304]. The preceding argument can be used to show that $HG \in OA^\circ$ if H and G are functions whose range is the normed ring N .

LEMMA 5.1. *If G is a function such that if $H \in OL^3$ on $[a, b]$ then $HG \in OA^\circ$ on $[a, b]$, then $G \in AZ$ on $[a, b]$.*

PROOF. It follows from Lemma 3.2 that $G \in OB^\circ$ on $[a, b]$. Suppose $G \notin AZ$ on $[a, b]$. Hence, there exists $\varepsilon > 0$ such that if D is a subdivision of $[a, b]$ then there exists a modified refinement L of D such that $\sum_{L(I)} |G| > \varepsilon$. There exist sequences $\{D_n\}_1^\infty$ and $\{L_n\}_1^\infty$ such that

- (1) D_n is a subdivision of $[a, b]$,
- (2) L_n is a modified refinement of D_n such that $\sum_{L_n(I)} |G| > \varepsilon$ and if $u \in L_n(I)$ then $|G(u)| < \varepsilon/n$, and
- (3) D_{n+1} is a refinement of L_n such that if $u \in L_n(I)$ then $u \notin D_{n+1}(I)$.

Let H be the function such that $H(u) = |G(u)|/G(u)$ if $u \in \bigcup_1^\infty L_n(I)$ and $G(u) \neq 0$, and $H(u) = 0$ otherwise. Observe that if $p \in [a, b]$ then there exists at most one number x such that $H(p, x) \neq 0$ and there exists at most one number y such that $H(y, p) \neq 0$. Hence, $H \in OL^3$.

Since $\int_a^b HG$ exists and $\varepsilon/4 > 0$, there exists a subdivision $D = \{x_q\}_0^n$ of $[a, b]$ such that if J and K are refinements of D then

$$\left| \sum_{J(I)} HG - \sum_{K(I)} HG \right| < \varepsilon/4.$$

Let J be a refinement of D such that if $u \in J(I)$ then $H(u)=0$, and hence, $\sum_{J(I)} HG=0$. Let $K=D \cup L_{8n} \cup D_{8n}$, and let U be the set such that $u \in U$ only if $u \in K(I)$ and u has a point of D as an endpoint. Let $V=K(I)-U$. Thus,

$$\begin{aligned} \varepsilon/4 &> \left| \sum_{K(I)} HG - \sum_{J(I)} HG \right| \\ &= \left| \left[\sum_V HG + \sum_U HG \right] + 0 \right| \geq \left| \sum_V HG \right| - 2n\varepsilon/8n \\ &\geq \left| \sum_{L_{8n}(I)} HG \right| - n\varepsilon/8n - \varepsilon/4 > \varepsilon - \varepsilon/8 - \varepsilon/4 > \varepsilon/2. \end{aligned}$$

This is a contradiction, and therefore, $G \in AZ$ on $[a, b]$.

THEOREM 5. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^3$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^3$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in AZ$ on $[a, b]$.*

THEOREM 6. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^4$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^4$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in AZ$ and $G \in OL^4$ on $[a, b]$.*

INDICATION OF PROOF. In these theorems it follows that (1) implies $G \in AZ$ by using Lemma 5.1. Further, in Theorem 6, G must be in OL^4 for (1) to imply (3) since $H \in OL^4$ does not imply that $H \in OL^3$. If (2) is true, then Lemma 3.2 implies that $G \in OB^\circ$ on $[a, b]$. Therefore, since (1) implies (3), it follows by using Theorem 1 that (2) implies (3). If $G \in AZ$, H is any bounded function and $\varepsilon > 0$, then there exists a subdivision D of $[a, b]$ such that if J is a refinement of D and U is the set such that $u \in U$ only if $u \in J(I)$ and contains a point of D , then

$$\left| \sum_{J(I)} HG \right| < \left| \sum_U HG \right| + \varepsilon.$$

By using this, in each theorem it can be shown that (3) implies (1), and thus, since $AZ \subseteq OB^\circ$, it follows from Theorem 1 that (3) implies (2).

THEOREM 7. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^{13}$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^{13}$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in OB^\circ$ on $[a, b]$.*

INDICATION OF PROOF. It follows from Lemma 3.2 that each of (1) and (2) implies (3). It follows easily that (3) implies (1). Hence, by using Theorem 1, it follows that (3) must also imply (2).

THEOREM 8. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^{14}$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^{14}$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in OB^\circ$ and $G \in OL^4$ on $[a, b]$.*

The proof is similar to Theorem 7.

THEOREM 9. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^{23}$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^{23}$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in OB^\circ$ on $[a, b]$.*

PROOF. By definition, $OL^{13} \subseteq OL^{23}$. Also, if $H \in OL^{23}$, $\varepsilon > 0$ and $a \leq x < y \leq b$, then there exist p and q such that $x < p < q < y$ and $|H(p, q)| < \varepsilon$. Hence, $OL^{23} \subseteq OL^{13}$. Therefore, since $OL^{13} = OL^{23}$, Theorem 9 is the same as Theorem 7.

THEOREM 10. *If G is a function, then the following are equivalent:*

- (1) *if $H \in OL^{24}$ on $[a, b]$, then $HG \in OA^\circ$ on $[a, b]$,*
- (2) *if $H \in OL^{24}$ on $[a, b]$, then $HG \in OM^\circ$ on $[a, b]$, and*
- (3) *$G \in OA^\circ$ and $G \in OB^\circ$ on $[a, b]$.*

PROOF. It follows from Theorem 2 that (3) implies (1) and (2). Further, it follows from Lemma 3.2 that each of (1) and (2) implies that $G \in OB^\circ$. If a function H is considered such that $H(x, y) \equiv 1$, it follows immediately that (1) implies $G \in OA^\circ$, and hence by using Theorem 1 that (2) implies $G \in OA^\circ$.

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DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85281