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TOEPLITZ OPERATORS AND DIFFERENTIAL EQUATIONS ON A HALF-LINE[1]

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ABSTRACT. Let $\mathscr{H}$ be a separable Hilbert space, let $A_0, A_1, \cdots, A_n$ denote bounded linear operators from $\mathscr{H}$ into $\mathscr{H}$, and let $\mathscr{D}$ represent the set of all functions in $L^2(0, \infty; \mathscr{H})$ whose first n derivatives belong to $L^2(0, \infty; \mathscr{H})$. Suppose further that the space $\mathscr{D}$ is equipped with an inner product inherited from $L^2(0, \infty; \mathscr{H})$. The main result of this note states that the differential operator

$$L = A_n \frac{d^n}{dt^n} + A_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + A_1 \frac{d}{dt} + A_0$$

acting on $\mathscr{D}$ is continuously invertible if and only if the operator

$$P(\sigma) = \sum A_k^* \sigma^k \qquad (0 \leqq k \leqq n)$$

acting on the Hilbert space $\mathscr{H}$ has a uniformly bounded inverse everywhere in the open half-plane $\operatorname{Re} \sigma < 0$.

Let $\mathscr{H}$ be a separable Hilbert space, and let $A_0, A_1, \cdots, A_n$ denote bounded linear operators from $\mathscr{H}$ into $\mathscr{H}$. In what follows we will obtain necessary and sufficient conditions to insure the continuous invertibility of the differential operator

$$L = A_n \frac{d^n}{dt^n} + A_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + A_1 \frac{d}{dt} + A_0 \tag{1}$$

acting on a dense manifold of the Hilbert space $L^2(0, \infty; \mathscr{H})$.

Our approach to this problem is based on the observation that L is unitarily equivalent to a generalized Toeplitz operator. The inversion theory of these operators was recently developed by Rabindranathan [**5**], who systematically extended the previous work of Widom [**7**], Devinatz [**1**], and Pousson [**4**]. Hereafter we will freely use the terminology, as well as some of the theory, contained in Rabindranathan's paper.

To expose the connections between L and a generalized Toeplitz operator, we first construct a special isometric mapping from $L^2(0, \infty; \mathscr{H})$

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onto $H^2(\mathscr{K})$, the Hardy space of $\mathscr{K}$-valued analytic functions defined on the interior of the unit disc. This is accomplished by taking the Laguerre functions

$$g_n(t) = \frac{1}{n!} \exp\left(\frac{t}{2}\right) \frac{d^n}{dx^n} [t^n \exp(-t)]$$

and then defining $J(g_n\varphi)=z^n\varphi$, $n=0, 1, \cdots$, for all $\varphi \in \mathscr{K}$. Since the Laguerre functions constitute an orthonormal basis for $L^2(0, \infty)$, the map J may be extended linearly as an isometry from $L^2(0, \infty; \mathscr{K})$ onto $H^2(\mathscr{K})$. In the scalar case where $\dim \mathscr{K}=1$, this mapping was employed by the author to study differentiability properties of exponential sums [**3**], and it also occurs in Rosenblum's earlier work on selfadjoint Toeplitz operators [**6**].

From the definition of a Laguerre function we deduce that

$$-2 \frac{dg_n}{dt}(t) = g_n(t) + 2 \sum g_k(t) \qquad (0 \leqq k \leqq n-1),$$

and a short computation reveals the important identity

$$2JDJ^{-1} = (T+I)(T-I)^{-1}, \tag{2}$$

where

$$(Tf)(z) = z^{-1}(f(z) - f(0))$$

and

$$(Dg)(t) = \lim_{h \to 0} h^{-1}(g(t+h) - g(t)).$$

This last limit is taken with respect to the norm topology on $L^2(0, \infty; \mathscr{K})$. Since the open unit disc comprises the point spectrum of T, the right side of (2) is well defined on the range of $T-I$, a set whose closure is all of $H^2(\mathscr{K})$ because it contains every vector-valued polynomial

$$p(z) = \varphi_n z^n + \varphi_{n-1} z^{n-1} + \cdots + \varphi_1 z + \varphi_0$$

with coefficients in $\mathscr{K}$.

If A is an operator from $\mathscr{K}$ into $\mathscr{K}$, we designate its natural extension $\hat{A}$ on $H^2(\mathscr{K})$ by writing

$$(\hat{A}f)(z) = \sum (A\varphi_n) z^n, \qquad n = 0, 1, 2, \cdots,$$

whenever $f(z)=\sum \varphi_n z^n$. Clearly $\hat{A}$ commutates with T, and the substitution of (2) into (1) yields

$$JLJ^{-1} = \sum 2^{-k} \hat{A}_k (T+I)^k (T-I)^{-k} \qquad (0 \leqq k \leqq n). \tag{3}$$

We infer from (3) that L has a bounded inverse if and only if the operator

$$S = (T - I)^{-n} \sum 2^{-k} \hat{A}_k (T + I)^k (T - I)^{n-k} \qquad (0 \leqq k \leqq n) \tag{4}$$

shares this property too.

A more effective method for determining the continuous invertibility of S may be obtained by examining the adjoint operator S^*. Since

$$(T^* f)(z) = z f(z) \quad \text{and} \quad (\hat{A}^* f)(z) = \sum (A^* \varphi_n) z^n,$$

a standard calculation involving the adjoint of a densely defined operator [**2**, p. 69] shows that

$$(S^* f)(z) = R(z) f(z) \tag{5}$$

where

$$R(z) = \sum 2^{-k} A_k^* (z + 1)^k (z - 1)^{-k} \qquad (0 \leqq k \leqq n). \tag{6}$$

According to a well-known result (Lemma 4.2 in [**5**]), S^* has a bounded inverse if and only if there exists an analytic Toeplitz operator $Q(z) = \sum Q_n z^n$, $n = 0, 1, \cdots$, defined on the interior of the unit disc such that

$$R(z) Q(z) = Q(z) R(z) = I \quad \text{and} \quad \sup_{|z|<1} \| Q(z) \| < \infty.$$

With this information at hand, it is possible to enunciate a simple invertibility criterion.

THEOREM. *Let $\mathscr{H}$ be a separable Hilbert space, let $A_0, A_1, \cdots, A_n$ denote bounded linear operators from $\mathscr{H}$ into $\mathscr{H}$, and let $\mathscr{D}$ represent the set of all functions in $L^2(0, \infty; \mathscr{H})$ whose first n derivatives lie in $L^2(0, \infty; \mathscr{H})$. Suppose further that $\mathscr{D}$ is endowed with the inner product inherited from $L^2(0, \infty; \mathscr{H})$. Then the differential operator*

$$L = A_n \frac{d^n}{dt^n} + A_{n-1} \frac{d^{n-1}}{dt^{n-1}} + \cdots + A_1 \frac{d}{dt} + A_0$$

acting on $\mathscr{D}$ has a bounded inverse if and only if the operator

$$P(\sigma) = \sum A_k^* \sigma^k \qquad (0 \leqq k \leqq n)$$

acting on the Hilbert space $\mathscr{H}$ has a uniformly bounded inverse everywhere in the open half-plane Re $\sigma < 0$.

PROOF. According to the arguments advanced before the derivation of (3), L has a bounded inverse if and only if the operator S enjoys the same property. It follows from elementary Hilbert space theory that L has a bounded inverse if and only if the adjoint operator S^* defined by (5) has a bounded inverse on $H^2(\mathscr{K})$. Moreover, since the linear fractional transformation $z=(2\sigma+1)(2\sigma-1)^{-1}$ maps the half-plane Re $\sigma<0$ onto the disc $|z|<1$, we see from Rabindranathan's lemma that S^* has a bounded inverse if and only if the operator-valued polynomial

$$R\left(\frac{2\sigma+1}{2\sigma-1}\right)=\sum A_k^*\sigma^k=P(\sigma)$$

has an analytic inverse which is uniformly bounded everywhere in the half-plane Re $\sigma<0$. But the inverse of an operator-valued polynomial is clearly analytic, and this completes the proof.

A special case of our theorem deserves attention because of its utility in the study of matrix differential equations.

COROLLARY. *When the dimension of $\mathscr{K}$ is finite, L has a bounded inverse if and only if the determinant of $P(\sigma)$ has no zeros in the closed left half-plane.*

One final remark should be made at this point: It seems quite probable that our techniques can be extended to cope with differential equations having *unbounded* coefficients. Some results in this direction are now being prepared for later publication.

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