

ON THE CONTINUITY OF BEST POLYNOMIAL APPROXIMATIONS

S. J. POREDA

ABSTRACT. Suppose f is a continuous complex valued function defined on a compact set E in the plane and $p_n(f, E)$ is the polynomial of degree n of best uniform approximation to f on E . If a polynomial q_n of degree n approximates f on E "almost" as well as $p_n(f, E)$, then q_n is "almost" $p_n(f, E)$. Sharp estimates, one for the real and one for the general case, are found for $\|q_n - p_n(f, E)\|_E$ in terms of the quantity $(\|f - q_n\|_E - \|f - p_n(f, E)\|_E)$, where $\|\cdot\|_E$ denotes the uniform norm on E .

1. Introduction. For a function f continuous on E , a compact set in the plane, let $\|f\|_E = \max_{z \in E} |f(z)|$. Also, for $n \in \mathbb{Z}^+$, let $p_n(f, E)$ denote the polynomial of degree n of best uniform approximation to f on E . A basic question that arises in the theory of best approximation is: If two continuous functions f_1 and f_2 are "close" on E , are their polynomials of best approximation $p_n(f_1, E)$ and $p_n(f_2, E)$ also "close" on E . More precisely, if $\{f_m\}_{m=1}^\infty$ is a sequence of continuous functions converging uniformly to f on E , does the sequence $\{p_n(f_m, E)\}_{m=1}^\infty$ converge uniformly to $p_n(f, E)$ on E (for each n) and if so, how rapid is the convergence.

The above problem can be stated in even greater generality. Suppose f is continuous on a compact set E , $n \in \mathbb{Z}^+$, $p_n(f, E) \equiv 0$, $\|f\|_E = 1$ and q_n is a polynomial of degree n for which $\|f - q_n\|_E \leq 1 + \varepsilon$, where $\varepsilon > 0$. Then, does $\|q_n\|_E$ approach zero as ε approaches zero and if so is there any relationship between their respective rates of convergence to zero. For example, is $\|q_n\|_E = O(\varepsilon^\beta)$ for some $\beta > 0$? We consider the real case first.

2. The real case. Our problem in the real case was settled in 1958 by G. Freud [4] who showed that $\|q_n\| = O(\varepsilon)$ where "O" depends only on E and f . His result also holds for approximation by generalized real valued polynomials (cf. Meinardus [1, p. 22]). We shall now state and prove Freud's result for ordinary polynomials and in the process describe

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in some way how "O" depends on E and f . Our proof shall also serve as a motivation for the corresponding proof in the complex case.

THEOREM 1. Suppose f is continuous and real valued on E , a compact subset of the real line, $p_n(f, E) \equiv 0$, $\|f\|_E = 1$ and q_n is a real polynomial of degree n for which $\|f - q_n\|_E < 1 + \varepsilon$ where $\varepsilon > 0$. It then follows that $\|q_n\|_E = O(\varepsilon)$, where "O" depends only on E and f . Furthermore this estimate is sharp for each n .

PROOF. By Chebyshev's Theorem [1, p. 20] there exist $n+2$ points $\{x_k\}_{k=1}^{n+2}$ in E such that

$$x_1 < x_2 < \cdots < x_{n+2}, \quad \text{and} \quad f(x_k) = -f(x_{k+1}) \\ \text{for } k = 1, 2, \cdots, n+1.$$

We may assume without loss of generality that $f(x_k) = (-1)^k$, $k = 1, 2, \cdots, n+2$. Let $w(x) = \prod_{k=1}^{n+2} (x - x_k)$, $M = \max_k |w'(x_k)|$ and $m = \min_k |w'(x_k)|$. We now claim that $|q_n(x_k)| < (n+1)M\varepsilon/m$, for $k = 1, 2, \cdots, n+2$.

If for some j , $1 \leq j \leq n+2$, $|q_n(x_j)| \geq (n+1)M\varepsilon/n$, then for some $l \neq j$, $1 \leq l \leq n+2$, $|q_n(x_l)| \geq \varepsilon$ and

$$(1) \quad \text{sign} \left[\frac{q_n(x_l)}{w'(x_l)} \right] = -\text{sign} \left[\frac{q_n(x_j)}{w'(x_j)} \right].$$

This follows, by Lagrange's interpolation formula since,

$$q_n(x) = w(x) \sum_{k=1}^{n+2} \frac{q_n(x_k)}{w'(x_k)(x - x_k)} = \left(\sum_{k=1}^{n+2} \frac{q_n(x_k)}{w'(x_k)} \right) x^{n+1} + \cdots,$$

and so $\sum_{k=1}^{n+2} q_n(x_k)/w'(x_k) = 0$, since q_n is a polynomial of degree n . Now by the hypothesis of our theorem, $|f(x_k) - q_n(x_k)| \leq 1 + \varepsilon$ for $k = 1, 2, \cdots, n+2$ and so

$$(2) \quad \begin{aligned} \text{sign}[q_n(x_j)] &= \text{sign}[f(x_j)] = \text{sign}[(-1)^j], \quad \text{and similarly,} \\ \text{sign}[q_n(x_l)] &= \text{sign}[f(x_l)] = \text{sign}[(-1)^l]. \end{aligned}$$

If we let, $l = t + j$, then $(-1)^l = (-1)^t(-1)^j$ and so by (2),

$$(3) \quad \text{sign}[q_n(x_l)] = \text{sign}[(-1)^t q_n(x_j)].$$

However, $\text{sign}[w'(x_k)]$ alternates on E and, in particular,

$$(4) \quad \text{sign}[w'(x_l)] = \text{sign}[(-1)^t w'(x_j)].$$

Thus by (3) and (4) we get

$$\operatorname{sign} \left[\frac{q_n(x_i)}{w'(x_i)} \right] = \operatorname{sign} \left[\frac{q_n(x_j)}{w'(x_j)} \right],$$

thus contradicting (1). Hence our claim follows.

Now since E is compact, the functions $\{w(x)/(x-x_k)\}_{k=1}^{n+2}$ are uniformly bounded on E , say by L , and so again using Lagrange's interpolation formula we can write

$$\begin{aligned} |q_n(x)| &= \left| \sum_{k=1}^{n+2} \frac{q_n(x_k)w(x)}{w'(x_k)(x-x_k)} \right| \leq \sum_{k=1}^{n+2} \left| \frac{q_n(x_k)w(x)}{w'(x_k)(x-x_k)} \right| \\ &\leq \frac{(n+2)(n+1)ML\varepsilon}{m^2}, \quad \text{for each } x \in E, \end{aligned}$$

where M and m are as before. Hence our theorem follows.

In order to demonstrate that the estimate $\|q_n\|_E = O(\varepsilon)$ is sharp for each n let $0 \leq x_1 < x_2 < \cdots < x_{n+2} < 1$ and $E = \{x_k\}_{k=1}^{n+2} \cup \{1\}$. Define the function f on E by setting $f(x_k) = (-1)^k$, $k=1, \dots, n+2$, and $f(1)=0$, and let $q_{n,\varepsilon}(x) = \varepsilon 2^n (x - \frac{1}{2})^n$. Then f , E and $q_{n,\varepsilon}$ satisfy the conditions of Theorem 1; however, $\|q_{n,\varepsilon}\|_E = |q_{n,\varepsilon}(1)| = \varepsilon$.

3. The complex case.

THEOREM 2. Suppose f is continuous on E , a compact set in the plane, $n \in \mathbb{Z}^+$, $p_n(f, E) \equiv 0$, $\|f\|_E = 1$ and q_n is a polynomial of degree n for which $\|f - q_n\|_E \leq 1 + \varepsilon$ where $1 > \varepsilon > 0$. It then follows that $\|q_n\|_E = O(\varepsilon^\beta)$, for every $\beta < \frac{1}{2}$, where " O " depends only on E , f and β . Furthermore this estimate is sharp for each n in that it is not in general true for $\beta = \frac{1}{2}$.

PROOF. By the Remez condition [3, p. 437] there exists m distinct points $\{z_k\}_{k=1}^m$ in E and m positive constants $\{\lambda_k\}_{k=1}^m$, $2n+3 \geq m \geq n+2$, such that

$$(i) \quad |f(z_k)| = 1 \quad \text{for } k = 1, 2, \dots, m, \quad \text{and}$$

$$(ii) \quad \sum_{k=1}^m \lambda_k \overline{f(z_k)} z_k^j = 0, \quad \text{for } j = 0, 1, \dots, n.$$

In particular,

$$(5) \quad \sum_{k=1}^m \lambda_k \overline{f(z_k)} q_n(z_k) = 0.$$

Set $\mu_k = f(z_k)$ for $k=1, 2, \dots, m$ and write $q_n(z_k) = (1 + \alpha_k)\mu_k + i\beta_k\mu_k$, $k=1, 2, \dots, m$, where the α_k 's and β_k 's are real. This can be done in a unique manner. By the hypothesis of our theorem we have that

$$|\alpha_k\mu_k + i\beta_k\mu_k| \leq 1 + \varepsilon, \quad \text{for } k = 1, 2, \dots, m,$$

so as a consequence

$$(6) \quad \alpha_k^2 + \beta_k^2 < (1 + \varepsilon)^2, \quad \text{and in particular,} \\ |\alpha_k| < (1 + \varepsilon).$$

With this notation, the expression (5) can be rewritten

$$\sum_{k=1}^m \lambda_k \bar{\mu}_k [(1 + \alpha_k) \mu_k + i \beta_k \mu_k] = 0,$$

or

$$\sum_{k=1}^m \lambda_k + \sum_{k=1}^m \lambda_k \alpha_k + i \sum_{k=1}^m \lambda_k \beta_k = 0.$$

Equating real parts yields

$$(7) \quad \sum_{k=1}^m \lambda_k \alpha_k = - \sum_{k=1}^m \lambda_k.$$

We now claim that for any $\beta < \frac{1}{2}$, $|q_n(z_k)| < \varepsilon^\beta$, for $k = 1, 2, \dots, m$, and all polynomials q_n satisfying the conditions of our theorem, if ε is sufficiently small. If for "sufficiently small" ε and some j , $1 \leq j \leq m$, $|q_n(z_j)| > \varepsilon^\beta$, it will then follow that

$$(8) \quad 1 + \alpha_j > (\varepsilon^{2\beta} - 2\varepsilon - \varepsilon^2)/2.$$

In order to demonstrate this we note that since $|q_n(z_j)| > \varepsilon^\beta$, we then have that $(1 + \alpha_j)^2 + \beta_j^2 > \varepsilon^{2\beta}$, and from (6) we have $(1 + \varepsilon)^2 > \alpha_j^2 + \beta_j^2$. Combining these two inequalities yields (8).

Now by (7) we have $\sum_{k=1, k \neq j}^m \lambda_k \alpha_k = - \sum_{k=1}^m \lambda_k - \alpha_j \lambda_j$, and so

$$\left| \sum_{k=1, k \neq j}^m \lambda_k \alpha_k \right| \geq \sum_{k=1}^m \lambda_k + \alpha_j \lambda_j.$$

Recalling (6) that $|\alpha_k| < (1 + \varepsilon)$ for $k = 1, 2, \dots, m$, we obtain

$$(1 + \varepsilon) \sum_{k=1, k \neq j}^m \lambda_k > \sum_{k=1}^m \lambda_k + \alpha_j \lambda_j,$$

and so

$$(9) \quad \varepsilon \sum_{k=1, k \neq j}^m \lambda_k > (1 + \alpha_j) \lambda_j > \lambda_j (\varepsilon^{2\beta} - 2\varepsilon - \varepsilon^2)/2.$$

This is impossible if ε is sufficiently small since $1 - 2\beta < 0$; hence our claim follows if we note that expression (9) does not depend on q_n . As in Theorem 1, we can complete our proof and show that $\|q_n\|_E = O(\varepsilon^\beta)$ by applying the Lagrange interpolation formula.

In order to demonstrate the sharpness of our result we construct for each $n \in \mathbb{Z}^+$ and each $M > 0$ a set E and a function f which satisfy the

conditions of our theorem and then construct for every sufficiently small ε a polynomial $q_{n,\varepsilon}(z)$ of degree n for which $\|f - q_{n,\varepsilon}\|_E \leq 1 + \varepsilon$ and such that $\|q_{n,\varepsilon}\|_E \leq M\varepsilon^{1/2}$.

We choose $n+2$ points $\{z_k\}_{k=1}^{n+2}$ such that if $w(z) = \prod_{k=1}^{n+2} (z - z_k)$,

$$(10) \quad \left| \sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right| < \sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|}, \quad \text{and let}$$

$$\tau_n(z) = w(z) \sum_{k=2}^{n+2} \frac{\mu w'(z_k) - |w'(z_k)|}{|w'(z_k)| w'(z_k)(z - z_k)},$$

where $\mu = (\sum_{k=2}^{n+2} 1/w'(z_k)) / (\sum_{k=2}^{n+2} 1/|w'(z_k)|)$. The polynomial τ_n is not identically constant and so there exists $z_0, z_0 \neq z_k, k=1, 2, \dots, n+2$, such that $|\tau_n(z_0)| > M+1$. Let $E = \{z_k\}_{k=0}^{n+2}$ and define a function f on E by setting $f(z_0) = 0$ and $f(z_k) = w'(z_k)/|w'(z_k)|$ for $k=1, 2, \dots, n+2$. It then follows [2] that $p_n(f, E) \equiv 0$ and $\|f\|_E = 1$.

Now for sufficiently small ε , let $\alpha = a(w'(z_1)/|w'(z_1)|) + ib(w'(z_1)/|w'(z_1)|)$, where $a = -(\varepsilon + \varepsilon^2)/2$ and $b = (\varepsilon - a^2)^{1/2}$. We define $q_{n,\varepsilon}$ by setting $q_{n,\varepsilon}(z) = (z - z_1)q_{n-1}(z) + \alpha$, where q_{n-1} is the polynomial of degree $n-1$ of best uniform approximation to the function $(f(z) - \alpha)/(z - z_1)$ on the set $\{z_k\}_{k=2}^{n+2}$ with respect to the weight function $|z - z_1|$. That is, q_{n-1} minimizes

$$\max_{2 \leq k \leq n+2} |z_k - z_1| |(f(z_k) - \alpha)/(z_k - z_1) - p_{n-1}(z_k)|$$

for all polynomials p_{n-1} of degree $n-1$. Set

$$\delta_n = \max_{2 \leq k \leq n+2} |z_k - z_1| |(f(z_k) - \alpha)/(z_k - z_1) - q_{n-1}(z_k)|.$$

Note that $\delta_n = \max_{2 \leq k \leq n+2} |f(z_k) - q_{n,\varepsilon}(z_k)|$, and so let us first show that $\delta_n \leq 1 + \varepsilon$. By applying the work [2] of Motzkin and Walsh, δ_n can be calculated explicitly, in fact

$$\begin{aligned} \delta_n &= \left| \left(\sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)} \right) / \left(\sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|} \right) \right| \\ &= \left| 1 - \alpha \left(\sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right) / \left(\sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|} \right) \right|. \end{aligned}$$

Now noting that $\sum_{k=1}^{n+2} 1/w'(z_k) = 0$ and by our choice of the z_k 's we can write

$$-\alpha \left(\sum_{k=2}^{n+2} \frac{1}{w'(z_k)} \right) / \left(\sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|} \right) = \alpha \sigma \left(\frac{|w'(z_1)|}{w'(z_1)} \right) = \sigma a + i \sigma b,$$

where $1 > \sigma \geq 0$. Thus

$$(11) \quad \delta_n^2 \leq (1 + \sigma a)^2 + (\sigma b)^2 < (1 + |a|)^2 + b^2 = (1 + \varepsilon)^2.$$

By our choice of the value $q_{n,\varepsilon}(z_1)$, a straightforward calculation yields

$$(12) \quad |f(z_1) - q_{n,\varepsilon}(z_1)| = 1 + \varepsilon.$$

Again by appealing to [2] we can calculate

$$q_{n-1}(z) = \frac{w(z)}{(z - z_1)} \left[\sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)(z - z_k)} - A_0 \sum_{k=2}^{n+2} \frac{1}{|w'(z_k)| (z - z_k)} \right],$$

where

$$A_0 = \left(\sum_{k=2}^{n+2} \frac{f(z_k) - \alpha}{w'(z_k)} \right) / \left(\sum_{k=2}^{n+2} \frac{1}{|w'(z_k)|} \right).$$

Now by substituting the given values for $f(z_k)$ we obtain $(z - z_1)q_{n-1}(z) = \alpha\tau_n(z)$. Hence,

$$\begin{aligned} \|q_{n,\varepsilon}\|_E &\geq |q_n(z_0)| = |(z_0 - z_1)q_{n-1}(z_0) + \alpha| \\ &\geq |\alpha\tau_n(z_0)| - |\alpha| \geq (M + 1)|\alpha| - |\alpha| = M\varepsilon^{1/2}. \end{aligned}$$

Also, $|f(z_0) - q_{n,\varepsilon}(z_0)| = |q_{n,\varepsilon}(z_0)| \leq 1 + \varepsilon$ if ε is sufficiently small, and so by (11) and (12) we get that $\|f - q_{n,\varepsilon}\|_E \leq 1 + \varepsilon$ and our example is complete.

4. Remark. As a consequence of Theorem 2, if a function f is continuous on a compact set E then for each $\beta < \frac{1}{2}$ and $n \in \mathbb{Z}^+$, there exists a least constant, M_n , such that if q_n is a polynomial of degree n for which $\|f - q_n\|_E \leq \|f - p_n(f, E)\|_E (1 + \varepsilon)$, where $0 < \varepsilon < 1$, then

$$\|p_n(f, E) - q_n\|_E \leq \|f - p_n(f, E)\|_E M_n \varepsilon^\beta.$$

Whether the sequence $\{M_n\}_{n=0}^\infty$ is bounded for each f and E remains open. A similar question can be posed in the real case.

REFERENCES

1. G. Meinardus, *Approximation of functions: Theory and numerical methods*, Springer, Berlin, 1964; English transl., Springer Tracts in Natural Philosophy, vol. 13, Springer-Verlag, New York, 1967. MR 31 #547; MR 36 #571.
2. T. S. Motzkin and J. L. Walsh, *On the derivative of a polynomial and Chebyshev approximation*, Proc. Amer. Math. Soc. 4 (1953), 76–87. MR 15, 701.
3. V. I. Smirnov and N. A. Lebedev, *Functions of a complex variable*, "Nauka", Moscow, 1964; English transl., M.I.T. Press, Cambridge, Mass., 1968. MR 30 #2152; MR 37 #5369.
4. G. Freud, *Eine Ungleichung für Tschebyscheffsche Approximations-polynome*, Acta Sci. Math. (Szeged) 19 (1958), 162–164. MR 21 #251.

DEPARTMENT OF MATHEMATICS, CLARK UNIVERSITY, WORCESTER, MASSACHUSETTS 01620