

A CHARACTERIZATION OF HILBERT MODULES¹

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ABSTRACT. For a proper H^* -algebra A , the relationship between $*$ -representations of A and Hilbert modules is discussed. It is shown that a Hilbert module structure can be defined on every essential $*$ -representation of A .

1. Introduction. A Hilbert module is a right module H over a proper H^* -algebra A , with a vector inner product (\cdot, \cdot) mapping $H \times H$ into the trace class of A , where (\cdot, \cdot) has certain properties analogous to a scalar valued inner product for a Hilbert space. P. P. Saworotnow [2] introduced the concept of a Hilbert module and showed that a linear space structure can be defined on H . Moreover, he proved that the vector inner product for H generates a Hilbert space inner product for H .

Of course, the right module structure means that an element a of A can be represented as an operator $\pi(a)$ on H . The Hilbert space structure induced on H is such that each $\pi(a)$ is linear with adjoint $\pi(a^*)$. In the terminology of [1], we have an essential $*$ -representation of the H^* -algebra A on the Hilbert space H , with the understanding that the representation is given by an antihomomorphism because of the right module structure.

The purpose of this paper is to show that the reverse process can be carried out. That is, given an essential $*$ -representation of a proper H^* -algebra A on a Hilbert space H , then a vector inner product can be defined for H such that H is a Hilbert module over A .

2. Definitions and notation. Throughout this paper A will denote a proper H^* -algebra, with trace class $\tau(A) = \{ab|a, b \in A\}$. A projection in A is a nonzero selfadjoint idempotent element of A . By an orthogonal projection base (OPB) for A is meant a maximal family of mutually orthogonal projections in A . If $\{e_\alpha\}$ is an OPB, then $A = \bigoplus \sum_\alpha A e_\alpha = \bigoplus \sum_\alpha e_\alpha A$.

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It is shown in [3] that $\tau(A)$ is a 2-sided ideal in A . In fact, $\tau(A)$ is a Banach algebra with respect to the trace norm defined below. The trace functional on $\tau(A)$ is denoted as $\text{tr}(\cdot)$. We have $\text{tr}(ab)=[a, b^*]=\sum_{\alpha} [abe_{\alpha}, e_{\alpha}]$ where $[\cdot, \cdot]$ is the Hilbert space inner product for A and $\{e_{\alpha}\}$ is an OPB. The trace norm $\tau(\cdot)$ on $\tau(A)$ is defined by $\tau(x)=\text{tr}([x])$, where $[x]^2=x^*x$.

DEFINITION. A right module H over A is called a Hilbert module if there exists a $\tau(A)$ -valued function (\cdot, \cdot) defined on $H \times H$ with the following properties:

- (1) $(f+g, h)=(f, h)+(g, h)$ for all $f, g, h \in H$.
- (2) $(f, g)^*=(g, f)$ for all $f, g \in H$.
- (3) $(f, ga)=(f, g)a$ for all $f, g \in H$ and each $a \in A$.
- (4) For each nonzero $f \in H$ there exists $a \neq 0$ in A such that $(f, f)=a^*a$.
- (5) $|\text{tr}(f, g)|^2 \leq \tau(f, f)\tau(g, g)$ for all $f, g \in H$.
- (6) H is complete with respect to the metric

$$d(f, g) = \{\text{tr}((f - g, f - g))\}^{1/2} = \tau((f - g, f - g))^{1/2}.$$

We can assume that a scalar multiplication is defined on H such that $\lambda(fa)=(\lambda f)a=f(\lambda a)$ for all complex numbers λ , $f \in H$ and $a \in A$. The equation $[f, g]=\text{tr}((f, g))$ ($f, g \in H$) defines the Hilbert space inner product referred to above. A detailed discussion of Hilbert modules may be found in [2].

3. The characterization theorem.

THEOREM. Let K be a Hilbert space which is a right module over A with the following properties:

- (1) $f(\lambda a)=\lambda(fa)=(\lambda f)a$ for all complex numbers λ , $f \in K$ and $a \in A$.
- (2) $[fa, g]=[f, ga^*]$ for all $f, g \in K$ and $a \in A$.
- (3) If $fa=0$ for all $a \in A$ then $f=0$.

Then K is a Hilbert module over A .

REMARKS. Notice that if K were a left module over A , the conditions of the theorem would be that A have an essential*-representation on K . Since any *-representation of a Banach *-algebra is continuous (Theorem 4.1.20 of [1]), we know there exists a constant m such that $\|fa\| \leq m\|f\|\|a\|$ for all $f \in K$ and $a \in A$.

LEMMA. Let $\{e_{\alpha}\}$ be an OPB for A . Then $f=\sum_{\alpha} fe_{\alpha}$, for all $f \in K$.

PROOF. Let J be any finite set of indices α . Then $f \rightarrow \sum_{\alpha \in J} fe_{\alpha}$ is clearly a projection operator on K , which implies that

$$\left\| \sum_{\alpha \in J} fe_{\alpha} \right\|^2 = \sum_{\alpha \in J} \|fe_{\alpha}\|^2 \leq \|f\|^2.$$

Thus the series of orthogonal terms fe_α converges to an element f' of K . Since each $a \in A$ can be written as $a = \sum_\alpha e_\alpha a$, by continuity we have $fa = f'a$ for all $a \in A$. Thus $f = f'$.

PROOF OF THE THEOREM. For $f \in K$, let T_f denote the map $a \mapsto fa$. Then T_f is a bounded operator from A to K and a simple calculation shows that $T_f^*(ga) = (T_f^*g)a$ for all $f, g \in K$ and $a \in A$. Let $b = T_f^*(f)$. Then $T_f^*T_f(x) = bx$ for all $x \in A$. This means that b is a positive element of A . If $\{e_\alpha\}$ is an OPB for A , then $\sum_\alpha [be_\alpha, e_\alpha] = \sum_\alpha [fe_\alpha, fe_\alpha] = \|f\|^2$. By Lemma 3 of [3], the convergence of the series implies that $b \in \tau(A)$. Since b is a positive element of $\tau(A)$, there exists a selfadjoint $c \in A$, such that $c^2 = b$.

We define a vector inner product for K by the equation $(f, g) = T_f^*(g)$ ($f, g \in K$). Since $T_f^*(g)$ can be expressed as a linear combination of four terms of the form $T_h^*(h)$, we have $(f, g) \in \tau(A)$. It is clear from the definition and the above discussion that conditions (1)–(4) of the Hilbert module definition are satisfied.

Finally, consider an OPB $\{e_\alpha\}$ for A . For $f, g \in K$, $\text{tr}((f, g)) = \sum_\alpha [T_f^*(g)e_\alpha, e_\alpha] = \sum_\alpha [ge_\alpha, fe_\alpha] = [g, f]$. Thus conditions (5) and (6) are satisfied, since (5) is the Cauchy-Schwartz inequality and (6) requires completeness of the Hilbert space norm for K .

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