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# COMPACTNESS IN TOPOLOGICAL TENSOR PRODUCTS AND OPERATOR SPACES

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ABSTRACT. Let $E$ and $F$ be Banach spaces, $E \otimes F$ their algebraic tensor product, and $E \otimes_\alpha F$ the completion of $E \otimes F$ with respect to a uniform crossnorm $\alpha \geqq \lambda$ (where $\lambda$ is the "least", and $\gamma$ the greatest, crossnorm). In §2 we characterize the relatively compact subsets of $E \otimes_\lambda F$ as those which, considered as spaces of operators from $E^*$ to $F$ and from $F^*$ to $E$, take the unit balls in $E^*$ and in $F^*$ to relatively compact sets in $F$ and $E$, respectively. In §3 we prove that if $T_1: E_1 \to E_2$ and $T_2: F_1 \to F_2$ are compact operators then $T_1 \otimes_\lambda T_2$ and $T_1 \otimes_\gamma T_2$ are each compact, and results concerning the problem for an arbitrary crossnorm $\alpha$ are also given. Schatten has characterized $(E \otimes_\alpha F)^*$ as a certain space of operators of "finite $\alpha$-norm". In §4 we show that a space of operators has such a representation if and only if its unit ball is weak operator compact.

1. **Introduction.** Throughout this paper $E$ and $F$ will denote Banach spaces, $E \otimes F$ their algebraic tensor product, and $E \otimes_\alpha F$ the completion of $E \otimes F$ with respect to a uniform crossnorm $\alpha \geqq \lambda$ (see [**6**] for notation and definitions used without other reference). Here $\lambda$ will denote the "least" crossnorm and $\gamma$ the greatest crossnorm [**6**].

In §2 we give a characterization of the relatively compact subsets of $E \otimes_\lambda F$. The characterization is given in operator terms, regarding $E \otimes_\lambda F$ as both a space of compact operators from $E^*$ to $F$ and one from $F^*$ to $E$.

If $T_1: E_1 \to E_2$ and $T_2: F_1 \to F_2$ are continuous linear operators then the tensor product maps $T_1 \otimes_\lambda T_2: E_1 \otimes_\lambda F_1 \to E_2 \otimes_\lambda F_2$ and $T_1 \otimes_\gamma T_2: E_1 \otimes_\gamma F_1 \to E_2 \otimes_\gamma F_2$ are each continuous. A question of considerable interest and importance in the theory of tensor products is the degree to which properties of the operators $T_1$ and $T_2$ carry over to the tensor product mappings (for results in this area see [**1**], [**2**], [**4**], and [**5**]). In §3 we show that if $T_1$ and $T_2$ are compact operators then $T_1 \otimes_\lambda T_2$ and $T_1 \otimes_\gamma T_2$ are also compact. A number of results concerning the more general situation in which $\alpha$ is any crossnorm for which $T_1 \otimes_\alpha T_2$ is continuous are also given.

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In §4 we consider the representation of Banach spaces of operators as duals of tensor products. Let $\mathscr{L}(E, F)$ denote the space of all continuous linear operators from $E$ to $F$ with norm given by $\|T\| = \sup_{\|x\|=1} \|Tx\|$. A linear subset $A(E, F)$ of $\mathscr{L}(E, F)$ which contains all finite-dimensional operators is called a *Banach space of operators* (or simply an *operator space*) if (i) $A(E, F)$ is a $B$-space under some norm which we call the $A$-norm on $A(E, F)$ and denote by $\|\cdot\|_A$, and (ii) if $T \in A(E, F)$ then $\|T\|_A \geqq \|T\|$, with equality holding for all one-dimensional operators. Schatten has shown that if $\alpha \geqq \lambda$ is a crossnorm on $E \otimes F$ then $(E \otimes_\alpha F)^*$ can be identified in a natural way with an operator space $A(E, F^*)$ [**6**]. In §4 we show that a given operator space $A(E, F^*)$ can be represented in the form $(E \otimes_\alpha F)^*$ for some $\alpha \geqq \lambda$ if and only if the unit ball in $A(E, F^*)$ is compact in the weak operator topology on $A(E, F^*)$.

2. In this section we will characterize the relatively compact subsets of $E \otimes_\lambda F$. To state this characterization in operator terms we recall that the space $E \otimes_\lambda F$ may be identified (in the obvious way) with the space of all compact operators from $E^*$ to $F$ which are continuous in the $w^*$-topology on $E^*$ and the weak topology on $F$ (of course, the same is true with the roles of $E$ and $F$ reversed) [**3**]. We will accordingly denote by $T$ an element of $E \otimes_\lambda F$, thinking of it as an operator from $E^*$ to $F$. Its adjoint $T^*: F^* \to E$ is associated with the same element of $E \otimes_\lambda F$, of course. Let $U^0$ be the unit ball in $E^*$ and $V^0$ the unit ball in $F^*$.

THEOREM 1. *A set $A \subset E \otimes_\lambda F$ is relatively compact if and only if each of the sets $A(U^0) = \{Tf \mid T \in A,\ f \in U^0\} \subset F$ and $A(V^0) = \{T^*g \mid T \in A,\ g \in V^0\} \subset E$ is relatively compact.*

PROOF. There is a natural embedding of $E \otimes_\lambda F$ into $C(U^0 \times V^0)$ (where $U^0 \times V^0$ has the product of the $w^*$-topologies on $U^0$ and $V^0$) obtained by embedding $E \to C(U^0)$, $F \to C(V^0)$ and $E \otimes_\lambda F \to C(U^0) \otimes_\lambda C(V^0) = C(U^0 \times V^0)$ [**2**]. Thus by Ascoli's theorem, to show $A \subset E \otimes_\lambda F$ is relatively compact we need only show that as a family of functions in $C(U^0 \times V^0)$ it is equicontinuous.

Let $\varepsilon > 0$ and $(f_0, g_0) \in U^0 \times V^0$. Since by assumption the sets $A(U^0)$ and $A(V^0)$ are relatively compact and hence totally bounded there exist $(y_i)_{i=1}^n \subset A(U^0)$ and $(x_j)_{j=1}^m \subset A(V^0)$ each of which is an $\varepsilon/6$-net in the respective sets.

If

$$N(f_0) = \{f \in U^0 \mid |\langle f, x_j \rangle - \langle f_0, x_j \rangle| < \varepsilon/6, j = 1, 2, \cdots, m\}$$

and

$$N(g_0) = \{g \in V^0 \mid |\langle g, y_i \rangle - \langle g_0, y_i \rangle| < \varepsilon/6, i = 1, 2, \cdots, n\}$$

are $w^*$-neighborhoods of $f_0$ and $g_0$ respectively then for any $(f, g) \in N(f_0) \times N(g_0)$ and any $T \in A$,

$$(2.1)\quad |\langle Tf, g\rangle - \langle Tf_0, g_0\rangle| \leqq |\langle Tf, g\rangle - \langle Tf, g_0\rangle| + |\langle Tf, g_0\rangle - \langle Tf_0, g_0\rangle|.$$

Now $(y_i)_{i=1}^n$ is an $\varepsilon/6$-net for $A(U^0)$ so there is a $y_k$ for which $\|Tf - y_k\| < \varepsilon/6$. Similarly there is an $x_p$ for which $\|T^*g_0 - x_p\| < \varepsilon/6$. Hence

$$(2.2)\quad \begin{aligned} |\langle Tf, g\rangle - \langle Tf, g_0\rangle| &\leqq |\langle Tf, g\rangle - \langle y_k, g\rangle| + |\langle y_k, g\rangle - \langle y_k, g_0\rangle| \\ &\quad + |\langle y_k, g_0\rangle - \langle Tf, g_0\rangle| \\ &< \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2. \end{aligned}$$

Similarly,

$$(2.3)\quad \begin{aligned} |\langle Tf, g_0\rangle - \langle Tf_0, g_0\rangle| &= |\langle f, T^*g_0\rangle - \langle f_0, T^*g_0\rangle| \\ &\leqq |\langle f, T^*g_0\rangle - \langle f, x_p\rangle| + |\langle f, x_p\rangle - \langle f_0, x_p\rangle| \\ &\quad + |\langle f_0, x_p\rangle - \langle f_0, T^*g_0\rangle| \\ &< \varepsilon/6 + \varepsilon/6 + \varepsilon/6 = \varepsilon/2. \end{aligned}$$

Combining (2.1), (2.2) and (2.3) we have that if $(f, g) \in N(f_0) \times N(g_0)$ and $T \in A$ then

$$|\langle Tf, g\rangle - \langle Tf_0, g_0\rangle| < \varepsilon.$$

By definition, then, $A$ is an equicontinuous subset of $C(U^0 \times V^0)$ and is therefore relatively compact.

Conversely, suppose $A \subset E \otimes_\lambda F$ is relatively compact. Then given $\varepsilon > 0$ there is an $\varepsilon/2$-net $(T_i)_{i=1}^n \subset A$. Since each $T_i$ is a compact operator from $E^*$ to $F$, the sets $T_i(U^0) \subset F$ are each relatively compact and hence totally bounded. Correspondingly, for each $1 \leqq i \leqq n$ there exists a sequence $(y_j^{(i)})_{j=1}^{m(i)} \subset T_i(U^0)$ which is an $\varepsilon/2$-net for $T_i(U^0)$.

We claim that the set $B = \{y_j^{(i)} \mid 1 \leqq i \leqq n, 1 \leqq j \leqq m(i)\} \subset A(U^0)$ is an $\varepsilon$-net for $A(U^0)$. For, if $T \in A$ then there exists a $1 \leqq k \leqq n$ for which $\|T - T_k\| < \varepsilon/2$. If $f \in U^0$ then $\|T_k f - Tf\| < \varepsilon/2$ and there is a $1 \leqq p \leqq m(k)$ for which $\|T_k f - y_p^{(k)}\| < \varepsilon/2$. It follows that $\|Tf - y_p^{(k)}\| < \varepsilon$ and $B$ is an $\varepsilon$-net for $T(U^0)$.

In the same way $A(V^0)$ is relatively compact and the proof is concluded.

Recall that if either $E^*$ or $F$ has the approximation property then $E^* \otimes_\lambda F$ can be identified with the space $K(E, F)$ of all compact operators from $E$ to $F$ [**2**].

COROLLARY. *If either $E^*$ or $F$ has the approximation property then a subset $A \subset K(E, F)$ is relatively compact $\Leftrightarrow$ each of the sets $A(U^0) = \{Tx \mid T \in A, x \in U = \text{unit ball of } E\}$ and $A(V^0) = \{T^*g \mid T \in A, g \in V^0 = \text{unit ball of } F^*\}$ are relatively compact.*

3. Let $T_1:E_1 \to E_2$ and $T_2:F_1 \to F_2$ be continuous linear operators. The operator defined on $E_1 \otimes E_2$ by

$$T_1 \otimes T_2\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} T_1 x_i \otimes T_2 y_i$$

is called the *tensor product operator*. If $\alpha$ is a $\otimes$-norm (in the terminology of Grothendieck [3]) then $T_1 \otimes T_2$ is continuous and may therefore be extended to the operator $T_1 \otimes_\alpha T_2 : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\alpha F_2$. In the study of tensor product spaces it is often necessary to know whether certain properties of the operators $T_1$ and $T_2$ are inherited by their tensor product. While this is often the case there are numerous exceptions ([2], [4], [5]). In this section we show that compactness of the operators $T_1$ and $T_2$ carries over to $T_1 \otimes_\alpha T_2$ for $\alpha = \lambda$ and $\alpha = \gamma$. Several remarks concerning the problem for arbitrary $\otimes$-norms are also given.

THEOREM 2. *Let $T_1:E_1 \to E_2$ and $T_2:F_1 \to F_2$ be compact linear operators. Then $T_1 \otimes_\lambda T_2 : E_1 \otimes_\lambda F_1 \to E_2 \otimes_\lambda F_2$ is compact.*

PROOF. Let $U^0$ and $V^0$ denote the unit balls in $E_2^*$ and $F_2^*$, respectively. Then, as is well known, we can embed $E_2$ in $l^\infty(U^0)$ and $F_2$ in $l^\infty(V^0)$ under the mappings $j_1 : E_2 \to l^\infty(U^0)$ and $j_2 : F_2 \to l^\infty(V^0)$.

Let $S_1 = j_1 \cdot T_1$ and $S_2 = j_2 \cdot T_2$. Then $S_1 \in K(E_1, l^\infty(U^0))$ and $S_2 \in K(F_1, l^\infty(V^0))$, so since each of $l^\infty(U^0)$ and $l^\infty(V^0)$ has the approximation property [2] there exist sequences $(P_n) \subset \mathscr{L}(E_1, l^\infty(U^0))$ and $(Q_n) \subset \mathscr{L}(F_1, l^\infty(V^0))$ of finite-dimensional operators whose norms are uniformly bounded and for which $\|S_1 - P_n\| \to^n 0$ and $\|S_2 - Q_n\| \to^n 0$. Then $(P_n \otimes_\lambda Q_n)$ is a sequence of finite-dimensional operators on $E_1 \otimes_\lambda F_1$ into $l^\infty(U^0) \otimes_\lambda l^\infty(V^0)$ for which

$$\begin{aligned}\|S_1 \otimes_\lambda S_2 - P_n \otimes_\lambda Q_n\| &\leqq \|S_1 \otimes_\lambda S_2 - P_n \otimes_\lambda S_2\| \\ &\quad + \|P_n \otimes_\lambda S_2 - P_n \otimes_\lambda Q_n\| \\ &\leqq \|S_2\| \, \|S_1 - P_n\| + \|P_n\| \, \|S_2 - Q_n\|.\end{aligned}$$

Since $\sup_n \|P_n\| < +\infty$ this last goes to zero with $n$, showing that $S_1 \otimes_\lambda S_2$ can be approximated arbitrarily closely in operator norm by finite-dimensional maps and hence is compact.

But $S_1 \otimes_\lambda S_2 = (j_1 \otimes_\lambda j_2) \cdot (T_1 \otimes_\lambda T_2)$ and it is well known that $j_1 \otimes_\lambda j_2$ is an isometry. Therefore $T_1 \otimes_\lambda T_2$ is also compact.

REMARK 1. It is clear from the proof of Theorem 2 that the theorem holds for any $\otimes$-norm $\alpha$ for which $j_1 \otimes_\alpha j_2$ is an isometry (or even an isomorphism).

REMARK 2. It is also clear from the proof of Theorem 2 that if both $T_1$ and $T_2$ can be approximated arbitrarily closely by finite-dimensional operators then the theorem holds for any $\otimes$-norm. This will be the case, then, if one of $E_1$ or $E_2$ and one of $F_1$ or $F_2$ has the approximation property.

THEOREM 3. *Let $T_1: E_1 \to E_2$ and $T_2: F_1 \to F_2$ be compact linear operators. Then $T_1 \otimes_\gamma T_2: E_1 \otimes_\gamma F_1 \to E_2 \otimes_\gamma F_2$ is compact.*

PROOF. It is clear we need only show $T_1 \otimes T_2: E_1 \otimes F_1 \to E_2 \otimes_\gamma F_2$ is compact. Let $(Z_k)$ be a $\gamma$-bounded sequence in $E_1 \otimes F_1$, say $Z_k = \sum_{i=1}^{n(k)} x_i^{(k)} \otimes y_i^{(k)}$ for $k=1, 2, \cdots$. We want to show that a subsequence of $(TZ_k)$ converges in $E_2 \otimes_\gamma F_2$.

To do this, note first that for any $k$ and $p$,

$$\|T_1 \otimes T_2(Z_k) - T_1 \otimes T_2(Z_p)\|_\gamma = \sup_{\|S\|=1; S \in \mathscr{L}(E_2, F_2^*)} |\langle Z_k - Z_p, T_2^* S T_1 \rangle|$$

by [6] and the definition of $T_1 \otimes T_2$. Hence we will be able to invoke Ascoli's theorem to obtain the desired result if the set $A = \{T_2^* S T_1 | S \in \mathscr{L}(E_2, F_2^*)$ with $\|S\| \leqq 1\}$ is relatively compact in $\mathscr{L}(E_1, F_1^*) = (E_1 \otimes_\gamma F_1)^*$ (since $(Z_k)$ is an equicontinuous subset of $C(A)$).

Let $\{T_2^* S_n T_1\}$ be a sequence in $A$. For any $n$, $m$,

$$\begin{aligned} \|T_2^* S_n T_1 - T_2^* S_m T_1\| &= \|T_2^*(S_n - S_m) T_1\| \\ &= \sup_{\|x\|=1, x \in E_1;\ \|y\|=1, y \in F_1} |\langle T_2^*(S_n - S_m) T_1 x, y \rangle| \\ &= \sup_{\|x\|=1, x \in E_1;\ \|y\|=1, y \in F_1} |\langle S_n - S_m T_1 x, T_2 y \rangle| . \end{aligned}$$

Again, since $(S_n)$ is an equicontinuous subset of $(E_2 \otimes_\gamma F_2)^*$, our theorem will be proved if the set $T_1(U) \otimes T_2(V) = \{T_1 x \otimes T_2 y | \ \|x\| \leqq 1, \ \|y\| \leqq 1\}$ is relatively compact in $E_2 \otimes_\gamma F_2$. But this is immediate since if $(T_1 x_n \otimes T_2 y_n)$ is a sequence in $T_1(U) \otimes T_2(V)$, then by virtue of the fact that each of $T_1$ and $T_2$ is compact there exist subsequences $(T_1 x_{n_k})$ and $(T_2 y_{n_k})$ which converge in $E_2$ and $F_2$, respectively, and hence for which $(T_1 x_{n_k} \otimes T_2 y_{n_k})$ also converges. The theorem is proved.

In view of Remark 2 following Theorem 2 it is, of course, very likely that the tensor product $T_1 \otimes_\alpha T_2$ of compact operators is compact for every $\otimes$-norm $\alpha$ (we continue to require $\alpha$ to be a $\otimes$-norm only to insure that $T_1 \otimes_\alpha T_2$ is continuous). The next theorem shows that the problem may be reduced to that of showing that one certain type of tensor product map is compact.

Let $E$ and $F$ be Banach spaces and $(f_n) \subset E^*$, $(g_n) \subset F^*$ sequences for which $\|f_n\| \to 0$, $\|g_n\| \to 0$. Define the mappings $T \in K(E, c_0)$ and $S \in K(F, c_0)$ by

$$Tx = (\langle f_n, x \rangle), \qquad Sy = (\langle g_n, y \rangle). \tag{3.1}$$

If $X$=closure of the range of $T$ and $Y$=closure of the range of $S$ then $T \in K(E, X)$ and $S \in K(F, Y)$.

THEOREM 4. *Let $\alpha$ be a $\otimes$-norm such that whenever $T \in K(E, X)$ and $S \in K(F, Y)$ are as above then $T \otimes_\alpha S: E \otimes_\alpha F \to X \otimes_\alpha Y$ is a compact operator. Then if $T_1: E_1 \to E_2$ and $T_2: F_1 \to F_2$ are compact operators the tensor product $T_1 \otimes_\alpha T_2$ is also compact.*

PROOF. It is known that since $T_1$ and $T_2$ are compact there exist sequences $(f_n) \subset E_1^*$ and $(g_n) \subset F_1^*$ such that $\|f_n\| \to 0$, $\|g_n\| \to 0$ and $\|T_1 x\| \leqq \sup_n |\langle f_n, x\rangle|$ for all $x \in E_1$, $\|T_2 y\| \leqq \sup_n |\langle g_n, y\rangle|$ for all $y \in F_1$ [7].

Thus $T_1$ may be factored as $T_1 = P_2 \cdot P_1$ where $P_1(x) = (\langle f_n, x\rangle) \in X \subset C_0$ for all $x \in E_1$ and $P_2: X \to E_2$ is defined by $P_2((\langle f_n, x\rangle)) = T_1 x$. Similarly $T_2$ can be factored as $T_2 = Q_2 \cdot Q_1$ where $Q_1(y) = (\langle g_n, y\rangle) \in Y \subset C_0$ for $y \in F_1$ and $Q_2: Y \to F_2$ is defined by $Q_2((\langle g_n, y\rangle)) = T_2 y$.

Since $T_1 \otimes_\alpha T_2 = (P_2 \otimes_\alpha Q_2) \cdot (P_1 \otimes_\alpha Q_1)$, where $P_1 \otimes_\alpha Q_1: E_1 \otimes_\alpha F_1 \to X \otimes_\alpha Y$ and $P_2 \otimes_\alpha Q_2: X \otimes_\alpha Y \to E_2 \otimes_\alpha F_2$, and by assumption $P_1 \otimes_\alpha Q_2$ is compact, we see that $T_1 \otimes_\alpha T_2$ is also compact.

REMARK 3. If one could show that the sets $X$ and $Y$ occurring in the above theorem have the approximation property then Theorem 4 together with Remark 2 would prove the result for all $\alpha$.

4. If $E$ and $F$ are Banach spaces and $\alpha$ a crossnorm on $E \otimes F$ then according to results of Schatten the space $(E \otimes_\alpha F)^*$ can be identified with the Banach space $A_\alpha(E, F^*)$ of all operators from $E$ to $F^*$ of "finite $\alpha$-norm" [**6**]. In this section we study the converse problem, that of determining when a given space of operators $A(E, F^*)$ is (under the identification given by Schatten) a space $A_\alpha(E, F^*) = (E \otimes_\alpha F)^*$ for some crossnorm $\alpha$.

The definition of a Banach space of operators $A(E, F^*)$ was given in §1. By the *weak operator topology* on $A(E, F^*)$ we mean the topology of pointwise convergence of nets in $A(E, F^*)$ on the set $E \times F$ (or, equivalently, on $E \otimes F \subset A(E, F^*)^*$).

We begin with a simple lemma. We emphasize that the embedding mentioned in the lemma (and denoted by the inclusion symbol) refers, as do similar embeddings throughout this section, to a very special embedding (namely, that of Schatten) which is explicitly defined in the proof of the lemma.

LEMMA. *If $A(E, F^*)$ is an operator space and $\alpha$ is defined on $E \otimes F$ by*

$$\alpha\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sup_{\|T\|_A = 1} \left| \sum_{i=1}^{n} \langle T x_i, y_i\rangle \right|,$$

*then $\alpha$ is a crossnorm and $E \otimes_\alpha F \subset A(E, F^*)^*$.*

PROOF. If $\sum_{i=1}^n x_i \otimes y_i \in E \otimes F$ then for $T \in A(E, F^*)$ the equation $\langle T, \sum_{i=1}^n x_i \otimes y_i \rangle = \sum_{i=1}^n \langle Tx_i, y_i \rangle$ identifies the tensor $\sum_{i=1}^n x_i \otimes y_i$ with a linear functional on $A(E, F^*)$. If $(T_m)$ is a sequence in $A(E, F^*)$ converging in $A$-norm to zero them $(T_m)$ also converges to zero in operator norm (since $\|T\|_A \geqq \|T\|$ for $T \in A(E, F^*)$), implying $\sum_{i=1}^n \langle T_m x_i, y_i \rangle \to^m 0$ and the functional $\sum_{i=1}^n x_i \otimes y_i$ is continuous on $A(E, F^*)$. Therefore we can identify in a canonical fashion the algebraic tensor product $E \otimes F$ with a linear subspace of $A(E, F^*)^*$. If we define

$$\alpha\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup_{\|T\|_A=1} \left|\sum_{i=1}^n \langle Tx_i, y_i \rangle\right|$$

then $\alpha$ is simply the norm induced on $E \otimes F$ considered as a subspace of $A(E, F^*)^*$ and so the embedding $E \otimes_\alpha F \to A(E, F^*)^*$ is an isometry.

Moreover if $x \otimes y \in E \otimes F$ then

$$\alpha(x \otimes y) = \sup_{\|T\|_A=1} \langle Tx, y \rangle \leqq \sup_{\|T\|=1} \langle Tx, y \rangle \leqq \|x\| \, \|y\|.$$

Conversely,

$$\alpha(x \otimes y) \geqq \sup_{\|f\|=\|g\|=1} \langle x \otimes y, f \otimes g \rangle = \|x\| \, \|y\|$$

since by assumption $A(E, F^*)$ contains all finite-dimensional operators and the $A$-norm of the one-dimensional operator $f \otimes g$ is equal to $\|f \otimes g\| = \|f\| \cdot \|g\|$. It follows that $\alpha$ is a crossnorm and the lemma is proved.

REMARK. Though we will not need the result in what follows, the crossnorm $\alpha$ defined above is $\geqq \lambda$. To see this note that

$$\alpha\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sup_{\|T\|_A=1} \left\langle \sum_{i=1}^n x_i \otimes y_i, T \right\rangle$$

$$\geqq \sup_{\|f\|=\|g\|=1} \left\langle \sum_{i=1}^n x_i \otimes y_i, f \otimes g \right\rangle$$

(as in the proof of the lemma)

$$= \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\lambda .$$

Using this lemma we now prove the converse to Schatten's theorem.

THEOREM 5. *Let $A(E, F^*)$ be an operator space. Then there is a crossnorm $\alpha$ for which $A(E, F^*) = (E \otimes_\alpha F)^*$ if and only if whenever $T \in \mathscr{L}(E, F^*)$ is the weak operator limit of a net $(T_\gamma)$ in $A(E, F^*)$ for which* $\sup_\gamma \|T_\gamma\|_A < +\infty$ *then $T \in A(E, F^*)$ and* $\|T\|_A \leqq \sup_\gamma \|T_\gamma\|_A$.

PROOF. Suppose $A(E, F^*)=A_\alpha(E, F^*)=(E \otimes_\alpha F)^*$ for some cross-norm $\alpha$ and $T \in \mathscr{L}(E, F^*)$ is the weak operator limit of the net $(T_\gamma)$ in $A(E, F^*)$ for which $\sup_\gamma \|T_\gamma\|_A = M < +\infty$.

If $\sum_{i=1}^n x_i \otimes y_i \in E \otimes_\alpha F$ and $\varepsilon > 0$ then there is a $\gamma_0$ for which

$$\left| \sum_{i=1}^n \langle Tx_i, y_i\rangle - \sum_{i=1}^n \langle T_{\gamma_0} x_i, y_i\rangle \right| < \varepsilon.$$

Hence

$$\begin{aligned} \left| \sum_{i=1}^n \langle Tx_i, y_i\rangle \right| &< \left| \sum_{i=1}^n \langle T_{\gamma_0} x_i, y_i\rangle \right| + \varepsilon \\ &\leqq \|T_{\gamma_0}\|_A \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\alpha + \varepsilon \leqq M \left\| \sum_{i=1}^n x_i \otimes y_i \right\|_\alpha + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we see that $T$ is a continuous linear functional on $E \otimes_\alpha F$ and is therefore by Schatten's theorem an element of $A_\alpha(E, F^*) = A(E, F^*)$. Moreover, $\|T\|_A \leqq M = \sup_\gamma \|T_\gamma\|_A$.

Conversely, suppose the operator space $A(E, F^*)$ has the stated property. Define $\alpha$ on $E \otimes F$ as in the preceding lemma. Then according to the lemma the canonical mapping $S: E \otimes_\alpha F \to A(E, F^*)^*$ described in the proof is an isometric isomorphism.

Let $T \in (E \otimes_\alpha F)^* = A_\alpha(E, F^*)$. By the Hahn-Banach theorem, $T$ extends to a functional $\tilde{T}$ in $A(E, F^*)^{**}$ with $\|T\|_{A_\alpha(E,F^*)} = \|\tilde{T}\|_{A(E,F^*)^{**}}$. Since the unit ball in $A(E, F^*)$ is weak*-dense in the unit ball of $A(E, F^*)^{**}$ there exists a net $(S_\gamma)$ in $A(E, F^*)$ for which $\sup_\gamma \|S_\gamma\|_A \leqq \|\tilde{T}\|$ and $\langle S_\gamma, x \otimes y\rangle \to \langle \tilde{T}, x \otimes y\rangle = \langle Tx, y\rangle$ for all $x \otimes y \in E \otimes F \subset A(E, F^*)^*$. It follows, then, that we actually have $(S_\gamma)$ weak operator convergent to $T$. By hypothesis, then, $T \in A_\alpha(E, F^*)$ and $\|T\|_A \leqq \sup_\gamma \|S_\gamma\| \leqq \|T\|_{A_\alpha(E,F^*)}$.

Thus we have shown that if $T \in A_\alpha(E, F^*)$ then $T \in A(E, F^*)$. The reverse inclusion is trivial since if $T \in A(E, F^*)$ then $T$ defines a continuous linear functional on $A(E, F^*)^*$ and hence also on $E \otimes_\alpha F \subset A(E, F^*)^*$. By Schatten's theorem $T \in A_\alpha(E, F^*)$ and $\|T\|_{A_\alpha(E,F^*)} \leqq \|T\|_A$. It follows that the spaces $A(E, F^*)$ and $A_\alpha(E, F^*)$ are isometrically isomorphic.

A more useful version of Theorem 5 is

COROLLARY. *Let $A(E, F^*)$ be an operator space. Then there is a cross-norm $\alpha \geqq \lambda$ such that $A(E, F^*) = (E \otimes_\alpha F)^*$ if and only if the unit ball in $A(E, F^*)$ is compact in the weak operator topology.*

PROOF. If $A(E, F^*) = (E \otimes_\alpha F)^*$ then the unit ball in $A(E, F^*)$ is compact in the $\sigma(A, (E, F^*), E \otimes_\alpha F)$-topology (by Alaoglu's theorem) and hence certainly weak operator compact since all $x \otimes y$ for $x \in E$, $y \in F$ are in $E \otimes_\alpha F$.

Conversely, suppose the unit ball in $A(E, F^*)$ is compact in the weak operator topology and $T \in \mathscr{L}(E, F^*)$ is the weak operator limit of the bounded net $(T_\gamma)$ in $A(E, F^*)$. Then by compactness of the unit ball there is an operator $S$ in the ball of radius $r = \sup_\gamma \|T_\gamma\|_A$ in $A(E, F^*)$ which is a weak operator cluster point of $(T_\gamma)$. But since $(T_\gamma)$ converges to $T$ in weak operator topology it must be that $S = T$ (the weak operator topology is Hausdorff), and hence, by Theorem 5, $A(E, F^*) = (E \otimes_\alpha F)^*$ for the crossnorm $\alpha$ constructed in that theorem.

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