ON RESTRICTED UNIQUENESS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. A uniqueness theorem is proved, for not necessarily Lipschitzian systems of ordinary differential equations, $y' = f$. This theorem compares with one of Okamura and Murakami, in that, here, at the expense of assuming a certain additional smoothness for f on open sets, no assumption is made regarding the existence of an auxiliary positive definite (Lyapunov) function. An example compares the relative applicability of the two theorems.

Let I denote the open real interval (a, b) , and let O be an open set in real n -dimensional Euclidean space E_n . Suppose $f: \bar{I} \times \bar{O} \rightarrow E_n$ is continuous. Consider the initial value problem

$$(1) \quad y'(x) = f(x, y(x)), \quad \text{for all } x \in I,$$

$$(2) \quad y(a) = y_a \in \bar{O}.$$

By a solution is meant a continuous function $u: \bar{I} \rightarrow \bar{O}$, which is differentiable, and satisfies (1), on I , and also satisfies (2). Clearly, if u is a solution, and f is continuous on $\bar{I} \times \bar{O}$, then the derivative u' , by (1), will be continuous on I .

The main concern of this paper is to prove a theorem, which will be compared with the results of Okamura [1] and Murakami [2] (see, also, Lakshmikantham and Leela [3]), which are based upon the existence of a certain positive-definite auxiliary (Lyapunov) function; see [2] or [3] for the actual statement. These results, stated in [2] and [3], are valid for the infinite dimensional initial value problem; to emphasize the simplicity of the present approach, however, only the finite dimensional case is considered here. The present theorem also compares to a result of Brauer and Sternberg [4], who obtain local uniqueness by assuming, as in [1] and [2], the existence of a positive-definite, Lyapunov-type auxiliary function.

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As pointed out by Murakami, uniqueness theorems which impose Lipschitz type inequalities on f may be inapplicable in some cases where the Lyapunov function, referred to above, may actually be constructed. In particular, he notices that the right-hand member of the scalar problem:

$$(3) \quad \begin{aligned} y'(x) &= f(x, y) = 1 + \sqrt{y}, & y \geq 0, \\ &= 1, & y < 0, \end{aligned}$$

$$(4) \quad y(0) = 0,$$

cannot satisfy the condition

$$\langle f(x, y) - f(x, z), y - z \rangle \leq M \|y - z\|^2,$$

which is required, for example, in the results of Browder [5] and Kato [6] (in the scalar case, this condition reduces to

$$(f(x, y) - f(x, z))(y - z) \leq M(y - z)^2);$$

however, it is shown that the required Lyapunov function exists, by displaying the complicated expression:

$$\begin{aligned} &\Phi(x, y, z) \\ &= [\sqrt{y} - \sqrt{z} - \log(1 + \sqrt{y}) + \log(1 + \sqrt{z})]^2, & y \geq 0, \quad z \geq 0, \\ (5) \quad &= [\sqrt{y} - \log(1 + \sqrt{y}) - \tfrac{1}{2}z]^2, & y \geq 0, \quad z < 0, \\ &= [\tfrac{1}{2}y - \sqrt{z} + \log(1 + \sqrt{z})]^2, & y < 0; \quad z \geq 0, \\ &= \tfrac{1}{4}[y - z]^2, & y < 0, \quad z < 0. \end{aligned}$$

Hence, the initial value problem (3), (4) has a unique solution. Note here that this particular construction is based on a specific knowledge of the solution to (3), (4), which is given, in implicit form, by $\sqrt{y} - \log(1 + \sqrt{y}) = \frac{1}{2}x$. It may be pointed out here that this uniqueness also follows as a trivial corollary to the theorem in [7], in which the proof is motivated by one of Fine [8]; see also Hartman [9, Theorem 8.1, p. 109].

In view of the complication of (5), it appears that, for even slightly more involved equations than (3), for example, $y' = \sqrt{y + x^2}$, there may be great difficulty in determining the required Lyapunov function (our attention was drawn, to this particular example, by Professor V. Lakshmikantham). The following theorem offers an alternative approach, which can be applied to many equations which do not satisfy a Lipschitz inequality, and which do not admit an obvious Φ in the Murakami result. In effect, the Murakami positive-definite, continuously differentiable function $\Phi(x, y, z)$ is replaced, simply and explicitly, by $V(x, y, z) = \frac{1}{2}\|f(x, y) - f(x, z)\|^2$, which may be merely positive semidefinite. Of

course, in this way, the need for any kind of construction of a suitable function $\Phi(x, y, z)$, as in (5), is absent.

With the above notation, assume $f \in C^1(I \times O)$, and define $F: I \times O \rightarrow E_{n+1}$ and $\phi: I \times O \rightarrow E_n$, by the equations $F(x, y) = (1, f(x, y))$, and $\phi(x, y) = (\langle \nabla f_1, F \rangle, \langle \nabla f_2, F \rangle, \dots, \langle \nabla f_n, F \rangle)$, where $f = (f_1, \dots, f_n)$, ∇ denotes the gradient in E_{n+1} , and $\langle \cdot, \cdot \rangle$ is the scalar product, in E_{n+1} .

THEOREM. *If $\langle f(x, y) - f(x, z), \phi(x, y) - \phi(x, z) \rangle \leq 0$, for all $(x, y), (x, z) \in I \times O$, where $\langle \cdot, \cdot \rangle$ is the scalar product in E_n , then there exists at most one solution u to (1), (2) on I , such that $u(x) \in O$ for $x \in I$.*

It is to be noted that the inequality in the hypothesis, when written in terms of the given real-valued functions $f_1, f_2, \dots, f_n$, is just

$$\sum_{i=1}^n [f_i(x, y) - f_i(x, z)] \left[\left(\frac{\partial f_i}{\partial x}(x, y) + \sum_{j=1}^n \frac{\partial f_i}{\partial y_j}(x, y) f_j(x, y) \right) - \left(\frac{\partial f_i}{\partial x}(x, z) + \sum_{j=1}^n \frac{\partial f_i}{\partial z_j}(x, z) f_j(x, z) \right) \right] \leq 0,$$

where $y = (y_1, y_2, \dots, y_n)$, $z = (z_1, z_2, \dots, z_n)$, and, where, using the customary notation, $\partial f_i / \partial y_j$ and $\partial f_i / \partial z_j$ denote the same partial derivative.

Before the proof, some remarks are in order. First, depending upon the choice of the open set O , there may be no solution satisfying $u(x) \in O$, for $x \in I$, or there may exist one or many solutions for which $u(x) \in O$, for $x \in I$. For example, if $n=1$, $a=0$, $b=1$, $y_0=0$, $y' = (|y|)^{1/2}$, and

$$I \times O = \{(x, y): 0 < x < 1, \frac{1}{4} < y < 1\},$$

then there is no solution in O . On the other hand, if

$$I \times O = \{(x, y): 0 < x < 1, 0 < y < 1\},$$

then $u(x) = \frac{1}{4}x^2$ for $0 \leq x \leq 1$ gives, as predicted by the theorem, the only solution for which $u(x) \in O$ for $0 < x < 1$. Finally, the main point here is that f may fail to satisfy a Lipschitz condition, due to its behavior near the boundary of $I \times O$; clearly, if the initial point y_a is an interior point of O , with all the other conditions being satisfied, then there is a locally unique solution, owing to the fact that the continuous differentiability of f then implies a Lipschitz condition in a suitably chosen neighborhood of (a, y_a) .

PROOF. Let $\|\cdot\|$ denote the Euclidean norm in E_n , and define

$$V(x) = \frac{1}{2} \|f(x, u(x)) - f(x, v(x))\|^2,$$

for $a \leq x \leq b$, where u is a solution, and v is a solution, in the sense of the theorem. Then, $V(a) = 0$, $V(x) \geq 0$ for $a \leq x \leq b$, and the derivative $V'(x)$

is given by

$$V'(x) = \langle f(x, u(x)) - f(x, v(x)), (d/dx)[f(x, u(x)) - f(x, v(x))] \rangle,$$

for $x \in I = (a, b)$. But, using the above definitions of F and ϕ , and the "chain rule" of the differential calculus, it follows that

$$(d/dx)[f(x, u(x)) - f(x, v(x))] = \phi(x, u(x)) - \phi(x, v(x)).$$

Hence, the hypothesis of the theorem then implies that $V'(x) \leq 0$ for $x \in I$. Thus, it must be true that $V(x) \equiv 0$ on I ; that is, $f(x, u(x)) \equiv f(x, v(x))$ on I . But, then, the differential equations satisfied, on I , by u and v , imply that $u'(x) = v'(x)$ for all $x \in I$, while the initial condition gives that $u(a) = v(a)$; therefore, $u(x) = v(x)$ for all $x \in I$. Q.E.D.

As a simple example, consider the scalar problem

$$\begin{aligned} y' &= f(x, y) = \sqrt{y} + g(x), & y \geq 0, 0 < x < 1, \\ &= g(x), & y < 0, 0 < x < 1, \\ y(0) &= 0, \end{aligned}$$

where $g \in C[0, 1]$, $C'(0, 1)$, and $g(x) \geq 0$ for $x \geq 0$. Then, with $I = (0, 1)$, and $I \times O = \{(x, y); x \in I \text{ and } y > 0\}$, one has, on $I \times O$:

$$\phi(x, y) = \langle \nabla f, (1, f) \rangle = g'(x) + \frac{1}{2}y^{-1/2}[y^{1/2} + g(x)].$$

Hence, for $(x, y), (x, z) \in I \times O$,

$$\begin{aligned} \langle f(x, y) - f(x, z), \phi(x, y) - \phi(x, z) \rangle &= \frac{1}{2}g(x)[y^{1/2} - z^{1/2}][y^{-1/2} - z^{-1/2}] \\ &= -\frac{1}{2}g(x)y^{-1/2}z^{-1/2}[y^{1/2} - z^{1/2}]^2 \leq 0, \end{aligned}$$

as required by the hypothesis of the theorem. The conclusion is that there is at most one solution $u(x)$ for which $u(x) > 0$ when $x > 0$. If $g(x) > 0$ for $x > 0$, then any local solution must be positive for $x > 0$; hence the local solution (given by Peano's theorem) must be the only positive solution in its local domain of definition.

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